07.2 ssresp.eig Linear algebraic eigenproblem

1 The linear algebraic **eigenproblem** can be simply stated. For $n \times n$ real matrix A, $n \times 1$ complex vector m, and $\lambda \in \mathbb{C}$, m is defined as an **eigenvector** of A if and only if it is nonzero and

$$Am = \lambda m \tag{1}$$

for some λ, which is called the corresponding **eigenvalue**. That is, m is an eigenvector of A if its linear transformation by A is equivalent to its scaling; i.e. an eigenvector of A is a vector of which A changes the length, but not the direction.

2 Since a matrix can have several eigenvectors and corresponding eigenvalues, we typically index them with a subscript; e.g. m_i pairs with λ_i.

Solving for eigenvalues

Eq. 1 can be rearranged:

$$(\lambda I - A)m = 0. \tag{2}$$

For a nontrivial solution for m,

$$\det(\lambda I - A) = 0, \tag{3}$$

which has as its left-hand-side a polynomial in λ and is called the **characteristic equation**. We define **eigenvalues** to be the roots of the characteristic equation.

Box 07 ssresp.1 eigenvalues and roots of the characteristic equation

If A is taken to be the linear state-space representation A, and the state-space model is converted to an input-output differential equation, the resulting ODE's "characteristic equation" would be identical to this matrix characteristic equation. Therefore, everything we already understand about the roots of the "characteristic equation" of an i/o ODE—especially that they govern the transient response and stability of a system—holds for a system's A-matrix eigenvalues.

3 Here we consider only the case of n distinct eigenvalues. For eigenvalues of (algebraic) multiplicity greater than one (i.e. repeated roots), see the discussion of Appendix 02.1 adv.eig.

Solving for eigenvectors

4 Each eigenvalue λ_i has a corresponding eigenvector $\mathbf{m}_i$. Substituting each λ_i into Eq. 2, one can solve for a corresponding eigenvector. It's important to note that an eigenvector is unique within a scaling factor. That is, if $\mathbf{m}_i$ is an eigenvector corresponding to λ_i, so is $3\mathbf{m}_i$.[3]

Example 07.2 ssresp.eig-1 **re: eigenproblem for a 2×2 matrix**

Let

$$A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of A.

[3] Also of note is that λ_i and $\mathbf{m}_i$ can be complex.

5 Several computational software packages can easily solve for eigenvalues and eigenvectors. See Lec. 07.3 ssresp.eigcomp for instruction for doing so in Matlab and Python.