07.4 ssresp.diag Diagonalizing basis

1 It is useful to transform a system's state vector x into a special basis that **diagonalizes**—leaves nonzero components along only the diagonal—the system's A-matrix. For systems with n distinct eigenvalues, to which we limit ourselves in this discussion,[4] this is always possible. In diagonalized form, it will be relatively easy to solve for the state transition matrix Φ.

Changing basis in the state equation

2 As with all basis transformations, the basis transformation we seek can be written

$$x = Px' \quad \Rightarrow x' = P^{-1}x, \tag{1}$$

where P is the **transformation matrix**, x is a representation of the state vector in the original basis, and x' is a representation of the state vector in the new basis.[5]

3 Substituting this transformation into the standard linear state-model equations yields the model

$$\dot{x}' = \underbrace{P^{-1}AP}_{A'}x' + \underbrace{P^{-1}B}_{B'}u \tag{2a}$$

$$y = \underbrace{CP}_{C'}x' + \underbrace{D}_{D'}u. \tag{2b}$$

Modal and eigenvalue matrices

4 Let a state equation have matrix A with n distinct eigenvalues (λ_i) and eigenvectors (m_i). Let the **eigenvalue matrix** Λ be defined as

[4] See Appendix 02.1 adv.eig for general considerations.

[5] We are being a bit fast-and-loose with terminology here: a vector is an object that does not change under basis transformation, only its components and basis vectors do. However, we use the common notational and terminological abuses.

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

5 Furthermore, let the **modal matrix** M be defined as

$$M = \begin{bmatrix} | & | & & | \\ \mathbf{m}_1 & \mathbf{m}_2 & \cdots & \mathbf{m}_n \\ | & | & & | \end{bmatrix} \tag{3}$$

Diagonalization of the state equation

6 Let the modal matrix M be the transformation matrix for our state-model. Then[6] $\mathbf{x}' = M^{-1}\mathbf{x}$.

7 The state equation becomes

$$\dot{\mathbf{x}}' = M^{-1}AM\mathbf{x}' + M^{-1}B\mathbf{u}. \tag{4}$$

The eigenproblem implies that

That is, $A' = \Lambda$! Recall that Λ is diagonal; therefore, we have **diagonalized** the state-space model. In full-form, the diagonalized model is

$$\dot{\mathbf{x}}' = \underbrace{\Lambda}_{A'}\mathbf{x}' + \underbrace{M^{-1}B}_{B'}\mathbf{u} \tag{5a}$$

$$\mathbf{y} = \underbrace{CM}_{C'}\mathbf{x}' + \underbrace{D}_{D'}\mathbf{u}. \tag{5b}$$

[6] As long as there are n distinct eigenvalues, M is invertible.

Computing the state transition matrix

8 Recall our definition of the state transition matrix $\Phi(t) = e^{At}$. Directly applying this to the diagonalized system of Eq. 5,

$$\Phi'(t) = e^{\Lambda t} \tag{6a}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}. \tag{6b}$$

In this last equality, we have used the **diagonal property** of the state transition matrix, defined in Lec. 07.1 ssresp.response.

9 Recall that the free response solution to the state equation is given by the initial condition and state transition matrix, so

$$\mathbf{x}'_{\text{fr}}(t) = \Phi'(t)\mathbf{x}'(0) \tag{7a}$$

$$= x'_1(0)e^{\lambda_1 t} + x'_2(0)e^{\lambda_2 t} + \cdots + x'_n(0)e^{\lambda_n t} \tag{7b}$$

where the initial conditions are $\mathbf{x}'(0) = M^{-1}\mathbf{x}(0)$. We have completely decoupled each state's free response, one of the remarkable qualities of the diagonalized system.

10 At this point, one could simply solve the diagonalized system for $\mathbf{x}'(t)$, then convert the solution to the original basis with $\mathbf{x}(t) = M\mathbf{x}'(t)$.

11 Sometimes, we might prefer to transform the diagonalized-basis state transition matrix into the original basis. The following is a derivation of that transformation.

12 Beginning with the free response solution in the diagonalized-basis and transforming the equation into the original basis, we find an expression for the original state transition matrix, as follows.

$$\begin{aligned}
\mathbf{x}'_{\text{fr}}(t) &= \Phi'(t)\mathbf{x}'(0) \quad \Rightarrow \\
M^{-1}\mathbf{x}_{\text{fr}}(t) &= \Phi'(t)M^{-1}\mathbf{x}(0) \quad \Rightarrow \\
\mathbf{x}_{\text{fr}}(t) &= \underbrace{M\Phi'(t)M^{-1}}_{\Phi(t)}\mathbf{x}(0).
\end{aligned}$$

This last expression is just the free response solution in the original basis, so we can identify

$$\Phi(t) = M\Phi'(t)M^{-1}. \tag{8}$$

This is a powerful result. Eq. 8 is the preferred method of deriving the state transition matrix for a given system. The eigenvalues give Φ' and the eigenvectors give M.

Example 07.4 ssresp.diag-1

re: state free response

For the state equation

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

find the state's free response to initial condition $x(0) = \begin{bmatrix} 2 & -1 \end{bmatrix}^\top$.