08.4 thermoflu.dam State-space model of a hydroelectric dam

1 Consider the microhydroelectric dam of Example 08.3 thermoflu.flutrans-1. We derived the linear graph of Fig. dam.1. In this lecture, we will derive a state-space model for the system—specifically, a state equation.

Normal tree, order, and variables

2 Now, we define a **normal tree** by overlaying it on the system graph in Fig. dam.1. There are six independent energy storage elements, making it a sixth-order ($n = 6$) system. We define the state vector to be

$$x = \begin{bmatrix} P_{C_1} & P_{C_2} & Q_{L_1} & \Omega_J & i_{L_2} & v_{C_3} \end{bmatrix}^\top. \tag{1}$$

The input vector is defined as $u = \begin{bmatrix} Q_s & P_{s1} & P_{s2} \end{bmatrix}^\top$.

Elemental equations

3 Yet to be encountered is a turbine's transduction. A simple model is that the torque T_2 is proportional to the flowrate Q_1, which are both

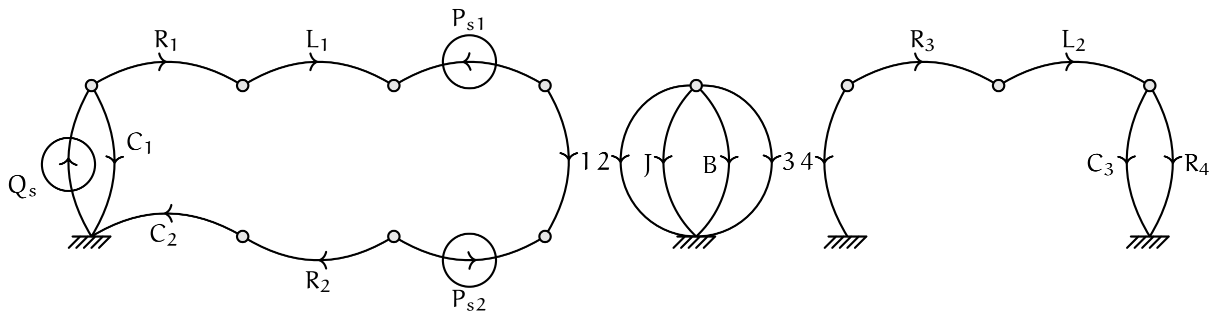

Figure dam.1: a linear graph for a microhydroelectric dam.

through-variables, making it a **transformer**, so

$$T_2 = -\alpha Q_1 \quad \text{and} \quad \Omega_2 = \frac{1}{\alpha} P_1, \tag{2}$$

where α is the **transformer ratio**.

4 The other elemental equations have been previously encountered and are listed, below.

el.	elemental eq.	el.	elemental eq.	el.	elemental eq.
C_1	$\frac{dP_{C_1}}{dt} = \frac{1}{C_1} Q_{C_1}$	C_3	$\frac{dv_{C_3}}{dt} = \frac{1}{C_3} i_{C_3}$	B	$\Omega_B = \frac{1}{B} T_B$
C_2	$\frac{dP_{C_2}}{dt} = \frac{1}{C_2} Q_{C_2}$	R_1	$P_{R_1} = Q_{R_1} R_1$	3	$i_4 = \frac{-1}{k_m} T_3$
L_1	$\frac{dQ_{L_1}}{dt} = \frac{1}{L_1} P_{L_1}$	R_2	$P_{R_2} = Q_{R_2} R_2$	4	$v_4 = k_m \Omega_3$
J	$\frac{d\Omega_J}{dt} = \frac{1}{J} T_J$	1	$T_2 = -\alpha Q_1$	R_3	$v_{R_3} = i_{R_3} R_3$
L_2	$\frac{di_{L_2}}{dt} = \frac{1}{L_2} v_{L_2}$	2	$\Omega_2 = \frac{1}{\alpha} P_1$	R_4	$i_{R_4} = \frac{1}{R_4} v_{R_4}$

Continuity and compatibility equations

5 Continuity and compatibility equations can be found in the usual way—by drawing contours and temporarily creating loops by including links in the normal tree. We proceed by drawing a table of all elements and writing a continuity equation for each branch of the normal tree and a compatibility equation for each link.

el.	eq.
C_1	$Q_{C_1} = Q_s - Q_{L_1}$
C_2	$Q_{C_2} = Q_{L_1}$
L_1	$P_{L_1} = -P_{R_1} + P_{C_1} - P_{C_2} +$
	$-P_{R_2} + P_{s2} - P_1 + P_{s1}$
J	$T_J = -T_2 - T_B - T_3$
L_2	$v_{L_2} = -v_{R_3} + v_4 - v_{C_3}$

el.	eq.
C_3	$i_{C_3} = i_{L_2} - i_{R_4}$
R_1	$Q_{R_1} = Q_{L_1}$
R_2	$Q_{R_2} = Q_{L_1}$
1	$Q_1 = Q_{L_1}$
2	$\Omega_2 = \Omega_J$

el.	eq.
B	$\Omega_B = \Omega_J$
3	$\Omega_3 = \Omega_J$
4	$i_4 = -i_{L_2}$
R_3	$i_{R_3} = i_{L_2}$
R_4	$v_{R_4} = v_{C_3}$

State equation

6 The system of equations composed of the elemental, continuity, and compatibility equations can be reduced to the state equation. There is a substantial amount of algebra required to eliminate those variables that are neither state nor input variables. Therefore, we use the Mathematica package *StateMint* (Devine **and** Picone, 2018). The resulting system model is:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u},$$

$$A = \begin{bmatrix} 0 & 0 & -1/C_1 & 0 & 0 & 0 \\ 0 & 0 & 1/C_2 & 0 & 0 & 0 \\ 1/L_1 & -1/L_1 & -(R_1 + R_2)/L_1 & -\alpha/L_1 & 0 & 0 \\ 0 & 0 & \alpha/J & -B/J & -k_m/J & 0 \\ 0 & 0 & 0 & k_m/L_2 & -R_3/L_2 & -1/L_2 \\ 0 & 0 & 0 & 0 & 1/C_3 & -1/(R_4 C_3) \end{bmatrix},$$

$$B = \begin{bmatrix} 1/C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/L_1 & 1/L_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7 The rub is estimating all these parameters.

8 The Mathematica notebook used above can be found in the source repository for this text.