

## 08.4 thermoflu.dam State-space model of a hydroelectric dam

1 Consider the microhydroelectric dam of [Example 08.3 thermoflu.flutrans-1](#). We derived the linear graph of [Fig. dam.1](#). In this lecture, we will derive a state-space model for the system—specifically, a state equation.

### Normal tree, order, and variables

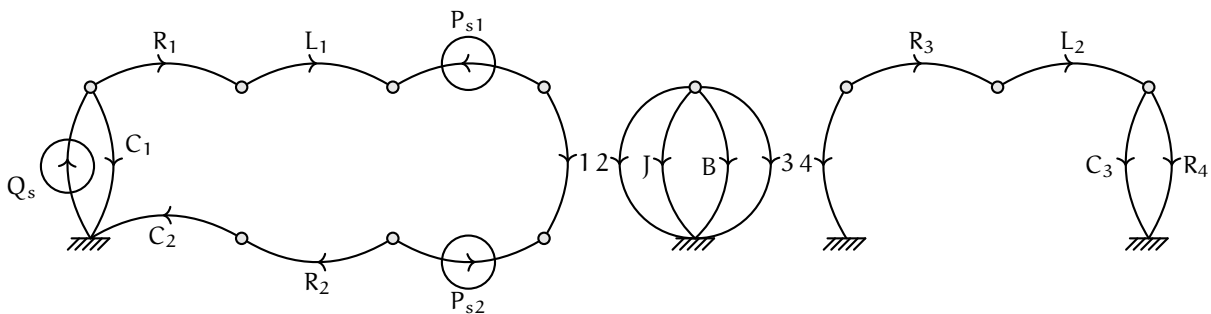
2 Now, we define a **normal tree** by overlaying it on the system graph in [Fig. dam.1](#). There are six independent energy storage elements, making it a sixth-order ( $n = 6$ ) system. We define the state vector to be

$$\mathbf{x} = [p_{C_1} \quad p_{C_2} \quad Q_{L_1} \quad \Omega_J \quad i_{L_2} \quad v_{C_3}]^T. \quad (1)$$

The input vector is defined as  $\mathbf{u} = [Q_s \quad P_{s1} \quad P_{s2}]^T$ .

### Elemental equations

3 Yet to be encountered is a turbine's transduction. A simple model is that the torque  $T_2$  is proportional to the flowrate  $Q_1$ , which are both



**Figure dam.1:** a linear graph for a microhydroelectric dam.

through-variables, making it a **transformer**, so

$$T_2 = -\alpha Q_1 \quad \text{and} \quad \Omega_2 = \frac{1}{\alpha} P_1, \quad (2)$$

where  $\alpha$  is the **transformer ratio**.

4 The other elemental equations have been previously encountered and are listed, below.

| el.   | elemental eq.                                 | el.   | elemental eq.                                 | el.   | elemental eq.                     |
|-------|---|-------|---|-------|-----------------------------------|
| $C_1$ | $\frac{dP_{C_1}}{dt} = \frac{1}{C_1} Q_{C_1}$ | $C_3$ | $\frac{dv_{C_3}}{dt} = \frac{1}{C_3} i_{C_3}$ | B     | $\Omega_B = \frac{1}{B} T_B$      |
| $C_2$ | $\frac{dP_{C_2}}{dt} = \frac{1}{C_2} Q_{C_2}$ | $R_1$ | $P_{R_1} = Q_{R_1} R_1$                       | 3     | $i_4 = \frac{-1}{k_m} T_3$        |
| $L_1$ | $\frac{dQ_{L_1}}{dt} = \frac{1}{L_1} P_{L_1}$ | $R_2$ | $P_{R_2} = Q_{R_2} R_2$                       | 4     | $v_4 = k_m \Omega_3$              |
| J     | $\frac{d\Omega_J}{dt} = \frac{1}{J} T_J$      | 1     | $T_2 = -\alpha Q_1$                           | $R_3$ | $v_{R_3} = i_{R_3} R_3$           |
| $L_2$ | $\frac{di_{L_2}}{dt} = \frac{1}{L_2} v_{L_2}$ | 2     | $\Omega_2 = \frac{1}{\alpha} P_1$             | $R_4$ | $i_{R_4} = \frac{1}{R_4} v_{R_4}$ |

### Continuity and compatibility equations

5 Continuity and compatibility equations can be found in the usual way—by drawing contours and temporarily creating loops by including links in the normal tree. We proceed by drawing a table of all elements and writing a continuity equation for each branch of the normal tree and a compatibility equation for each link.

| el.            | eq.  | el.            | eq.                           | el.            | eq.                   |
|----------------|--|----------------|-------------------------------|----------------|-----------------------|
| C <sub>1</sub> | $Q_{C_1} = Q_s - Q_{L_1}$  | C <sub>3</sub> | $i_{C_3} = i_{L_2} - i_{R_4}$ | B              | $\Omega_B = \Omega_J$ |
| C <sub>2</sub> | $Q_{C_2} = Q_{L_1}$  | R <sub>1</sub> | $Q_{R_1} = Q_{L_1}$           | 3              | $\Omega_3 = \Omega_J$ |
| L <sub>1</sub> | $P_{L_1} = -P_{R_1} + P_{C_1} - P_{C_2} +$<br>$-P_{R_2} + P_{s2} - P_1 + P_{s1}$ | R <sub>2</sub> | $Q_{R_2} = Q_{L_1}$           | 4              | $i_4 = -i_{L_2}$      |
| J              | $T_J = -T_2 - T_B - T_3$   | 1              | $Q_1 = Q_{L_1}$               | R <sub>3</sub> | $i_{R_3} = i_{L_2}$   |
| L <sub>2</sub> | $v_{L_2} = -v_{R_3} + v_4 - v_{C_3}$   | 2              | $\Omega_2 = \Omega_J$         | R <sub>4</sub> | $v_{R_4} = v_{C_3}$   |

### State equation

6 The system of equations composed of the elemental, continuity, and compatibility equations can be reduced to the state equation. There is a substantial amount of algebra required to eliminate those variables that are neither state nor input variables. Therefore, we use the Mathematica package *StateMint* (Devine and Picone, 2018). The resulting system model is:

$$\frac{dx}{dt} = Ax + Bu,$$

$$A = \begin{bmatrix} 0 & 0 & -1/C_1 & 0 & 0 & 0 \\ 0 & 0 & 1/C_2 & 0 & 0 & 0 \\ 1/L_1 & -1/L_1 & -(R_1 + R_2)/L_1 & -\alpha/L_1 & 0 & 0 \\ 0 & 0 & \alpha/J & -B/J & -k_m/J & 0 \\ 0 & 0 & 0 & k_m/L_2 & -R_3/L_2 & -1/L_2 \\ 0 & 0 & 0 & 0 & 1/C_3 & -1/(R_4 C_3) \end{bmatrix},$$

$$B = \begin{bmatrix} 1/C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/L_1 & 1/L_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7 The rub is estimating all these parameters.

8 The Mathematica notebook used above can be found in the [source repository](#) for this text.