08.5 thermoflu.fem Thermal finite element model

Example 08.5 thermoflu.fem-1

re: thermal finite element model

Figure fem.1: an insulated bar.

Consider the long homogeneous copper bar of Fig. fem.1, insulated around its circumference, and initially at uniform temperature. At time $t = 0$, the temperature at one end of the bar ($x = 0$) is increased by one Kelvin. We wish to find the dynamic variation of the temperature at any location x along the bar, at any time $t > 0$.

Construct a discrete element model of thermal conduction in the bar, for which the following parameters are given for its length `L` and diameter `d`.

```
L = 1; % m
d = 0.01; % m
```

Geometrical considerations

The cross-sectional area for the bar is as follows.

```
a = pi/4*d^2; % m^2 x-sectional area
```

Dividing the bar into `n` sections ("finite elements") such that we have length of each `dx` gives the following.

```
n = 100; % number of chunks
dx = L/n; % m ... length of chunk
```

Material considerations

The following are the material properties of copper.

```
cp = 390; % SI ... specific heat capacity
rho = 8920; % SI ... density
ks = 401; % SI ... thermal conductivity
```

Lumping

From the geometrical and material considerations above, we can develop a lumped thermal resistance `R` and thermal capacitance `c` of each cylindrical section of the bar of length `dx`. From Eq. 6 and Eq. 4, these parameters are as follows.

```
R = dx/(ks*a); % thermal resistance
dV = dx*a; % m^3 ... section volume
dm = rho*dV; % kg ... section mass
c = dm*cp; % section volume
```

Linear graph model

The linear graph model is shown in Fig. fem.2 with the corresponding normal tree overlayed.

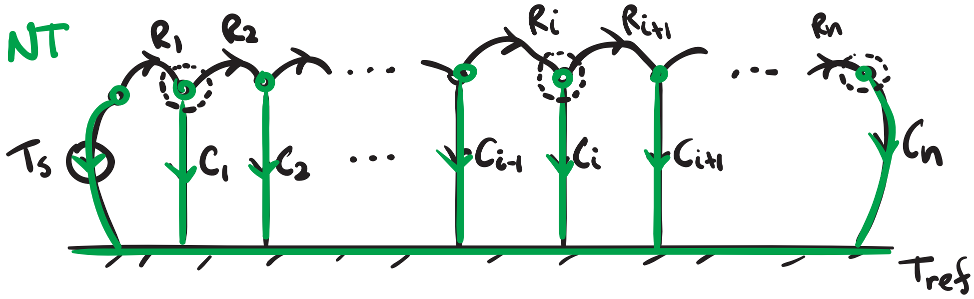

Figure fem.2: a linear graph of the insulated bar.

State-space model

The state variables are clearly the temperatures of C_i: $T_{C_1}, \cdots, T_{C_n}$. Therefore, the order of the system is n.
The state, input, and output variables are

$$x = \begin{bmatrix} T_{C_1} \cdots T_{C_n} \end{bmatrix}^\top, \quad u = \begin{bmatrix} T_S \end{bmatrix}, \text{ and } \quad y = x. \tag{1}$$

Elemental, continuity, and compatibility equations Consider the elemental, continuity, and compatibility equations, below, for the first, a middle, and the last elements. The following makes the assumption of homogeneity, which yields $R_i = R$ and $C_i = C$.

element	elemental eq.	continuity eq.	compatibility eq.
C_1	$\dot{T}_{C_1} = \frac{1}{C} q_{C_1}$	$q_{C_1} = q_{R_1} - q_{R_2}$	
R_1	$q_{R_1} = \frac{1}{R} T_{R_1}$		$T_{R_1} = T_S - T_{C_1}$
C_i	$\dot{T}_{C_i} = \frac{1}{C} q_{C_i}$	$q_{C_i} = q_{R_i} - q_{R_{i+1}}$	
R_i	$q_{R_i} = \frac{1}{R} T_{R_i}$		$T_{R_i} = T_{C_{i-1}} - T_{C_i}$
C_n	$\dot{T}_{C_n} = \frac{1}{C} q_{C_n}$	$q_{C_n} = q_{R_n}$	
R_n	$q_{R_n} = \frac{1}{R} T_{R_n}$		$T_{R_n} = T_{C_{n-1}} - T_{C_n}$

Deriving the state equations for sections 1, i, and n For each of the first, a representative middle, and the last elements, we can derive the state equation with relatively few substitutions, as follows.

$$
\begin{aligned}
\dot{T}_{C_1} &= \frac{1}{C} q_{C_1} && \text{(elemental)} \\
&= \frac{1}{C}(q_{R_1} - q_{R_2}) && \text{(continuity)} \\
&= \frac{1}{RC}(T_{R_1} - T_{R_2}) && \text{(elemental)} \\
&= \frac{1}{RC}(T_S - T_{C_1} - T_{C_1} + T_{C_2}) && \text{(compatibility)} \\
&= \frac{1}{RC}(T_S - 2T_{C_1} + T_{C_2}). \\
\dot{T}_{C_i} &= \frac{1}{C} q_{C_i} && \text{(elemental)} \\
&= \frac{1}{C}(q_{R_i} - q_{R_{i+1}}) && \text{(continuity)} \\
&= \frac{1}{RC}(T_{R_i} - T_{R_{i+1}}) && \text{(elemental)} \\
&= \frac{1}{RC}(T_{C_{i-1}} - 2T_{C_i} + T_{C_{i+1}}). && \text{(compatibility)} \\
\dot{T}_{C_n} &= \frac{1}{C} q_{C_n} && \text{(elemental)} \\
&= \frac{1}{C} q_{R_n} && \text{(continuity)} \\
&= \frac{1}{RC} T_{R_n} && \text{(elemental)} \\
&= \frac{1}{RC}(T_{C_{n-1}} - T_{C_n}). && \text{(compatibility)}
\end{aligned}
$$

Let $\tau = RC$. The A and B matrices are, then

$$A = \begin{bmatrix} -2/\tau & 1/\tau & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1/\tau & -2/\tau & 1/\tau & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & \ddots & \ddots & \ddots & & & & & \\ \vdots & & & & 1/\tau & -2/\tau & 1/\tau & & & & \vdots \\ & & & & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1/\tau & -2/\tau & 1/\tau \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1/\tau & -1/\tau \end{bmatrix}$$

$$B = \begin{bmatrix} 1/\tau \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} . \tag{2}$$

The outputs are the states: $y = x$. Or, in standard form with identity matrix I, the matrices are:

$$C = I_{n\times n} \quad \text{and} \quad D = 0_{n\times 1}. \tag{3}$$

Simulation of a step response

Define the `A` matrix.

```
A = zeros(n);
% first row
A(1,1) = -2/(R*c);
A(1,2) = 1/(R*c);
% last row
A(n,n-1) = 1/(R*c);
A(n,n) = -1/(R*c);
% middle rows
for i = 2:(n-1)
  A(i,i-1) = 1/(R*c);
  A(i,i) = -2/(R*c);
  A(i,i+1) = 1/(R*c);
end
```

Now define `B`, `C`, and `D`.

```
B = zeros([n,1]);
B(1) = 1/(R*c);
C = eye(n);
D = zeros([n,1]);
```

Create a state-space model.

```
sys = ss(A,B,C,D);
```

Simulate a unit step in the input temperature.

```
Tmax = 1200; % sec ... final sim time
t = linspace(0,Tmax,100);
y = step(sys,t);
```

Plot the step response To prepare for creating a 3D plot, we need to make a grid of points.

```
x = dx/2:dx:(L-dx/2);
[X,T] = meshgrid(x,t);
```

Now we're ready to plot. The result is shown in Fig. fem.3.

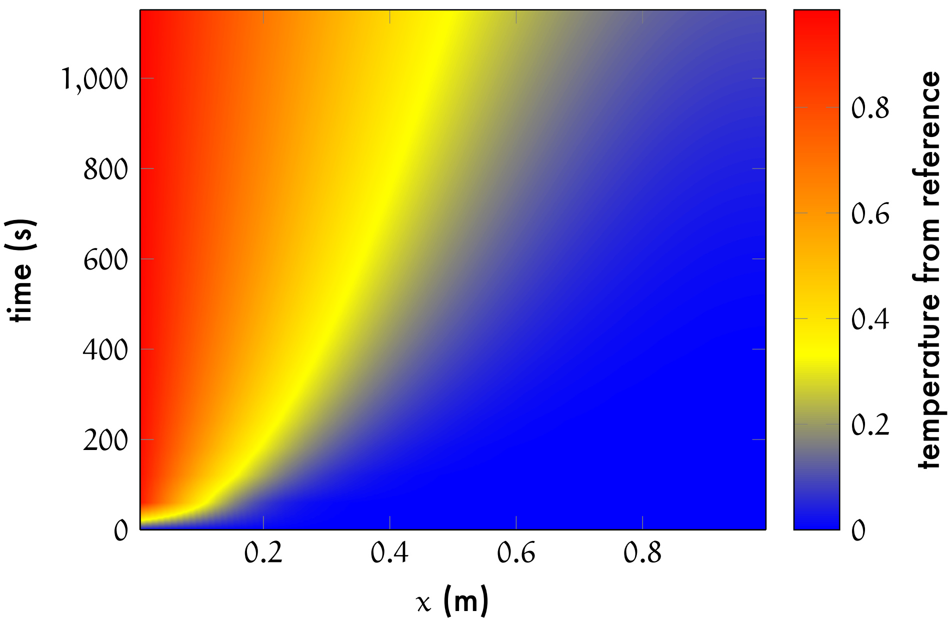

Figure fem.3: spatiotemporal thermal response.

```
figure
contourf(X,T,y)
shading(gca,'interp')
xlabel('x')
ylabel('time')
zlabel('temp (K)')
```