09.1 four.series Fourier series

1 Fourier series are mathematical series that can represent a periodic signal as a sum of sinusoids at different amplitudes and frequencies. They are useful for solving for the response of a system to periodic inputs. However, they are probably most important *conceptually*: they are our gateway to thinking of signals in the **frequency domain**—that is, as functions of *frequency* (not time). To represent a function as a Fourier series is to **analyze** it as a sum of sinusoids at different frequencies[1] ω_n and amplitudes a_n. Its **frequency spectrum** is the functional representation of amplitudes a_n versus frequency ω_n.

2 Let's begin with the definition.

Definition 09 four.1: Fourier series: trigonometric form

The *Fourier analysis* of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} y(t)dt \tag{1}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(\omega_n t)dt \tag{2}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(\omega_n t)dt. \tag{3}$$

The *Fourier synthesis* of a periodic function $y(t)$ with analysis components a_n and b_n corresponding to ω_n is

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t). \tag{4}$$

3 Let's consider the complex form of the Fourier series, which is analogous to Definition 09 four.1. It may be helpful to review Euler's formula(s) – see Appendix 04.1 com.euler.

[1] It's important to note that the symbol ω_n, in this context, is not the natural frequency, but a frequency indexed by integer n.

Definition 09 four.2: Fourier series: complex form

The *Fourier analysis* of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$c_{\pm n} = \frac{1}{T} \int_{-T/2}^{T/2} y(t) e^{-j\omega_n t} dt. \tag{5}$$

The *Fourier synthesis* of a periodic function $y(t)$ with analysis components c_n corresponding to ω_n is

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}. \tag{6}$$

4 We call the integer n a **harmonic** and the frequency associated with it,

$$\omega_n = 2\pi n/T, \tag{7}$$

the **harmonic frequency**. There is a special name for the first harmonic ($n = 1$): the **fundamental frequency**. It is called this because all other frequency components are integer multiples of it.

5 It is also possible to convert between the two representations above.

Definition 09 four.3: Fourier series: converting between forms

The complex Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and a_n and b_n as defined above,

$$c_{\pm n} = \frac{1}{2} \left(a_{|n|} \mp j b_{|n|} \right) \tag{8}$$

The sinusoidal Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and c_n as defined above,

$$a_n = c_n + c_{-n} \text{ and} \tag{9}$$

$$b_n = j\left(c_n - c_{-n}\right). \tag{10}$$

6 The **harmonic amplitude** C_n is

$$C_n = \sqrt{a_n^2 + b_n^2} \tag{11}$$

$$= 2\sqrt{c_n c_{-n}}. \tag{12}$$

A **magnitude line spectrum** is a graph of the harmonic amplitudes as a function of the harmonic frequencies. The **harmonic phase** is

$$\begin{aligned}\theta_n &= -\operatorname{arctan}_2(b_n, a_n) && \text{(see Appendix 01.2 math.trig)} \\ &= \operatorname{arctan}_2(\operatorname{Im}(c_n), \operatorname{Re}(c_n)). && (13)\end{aligned}$$

7 The illustration of Fig. series.1 shows how sinusoidal components sum to represent a square wave. A line spectrum is also shown.

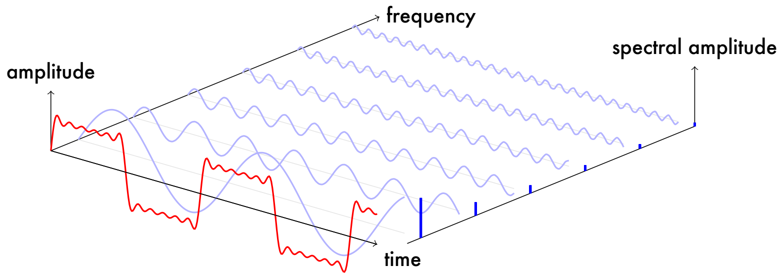

Figure series.1: a partial sum of Fourier components of a square wave shown through time and frequency. The spectral amplitude shows the amplitude of the corresponding Fourier component.

8 Let us compute the associated spectral components in the following example.

Example 09.1 four.series-1

re: Fourier series analysis: line spectrum

Compute the first five harmonic amplitudes that represent the line spectrum for a square wave in the figure above.