

## 09.1 four.series Fourier series

**1** Fourier series are mathematical series that can represent a periodic signal as a sum of sinusoids at different amplitudes and frequencies. They are useful for solving for the response of a system to periodic inputs. However, they are probably most important *conceptually*: they are our gateway to thinking of signals in the **frequency domain**—that is, as functions of *frequency* (not time). To represent a function as a Fourier series is to **analyze** it as a sum of sinusoids at different frequencies<sup>1</sup>  $\omega_n$  and amplitudes  $a_n$ . Its **frequency spectrum** is the functional representation of amplitudes  $a_n$  versus frequency  $\omega_n$ .

**2** Let's begin with the definition.

### Definition 09 four.1: Fourier series: trigonometric form

The *Fourier analysis* of a periodic function  $y(t)$  is, for  $n \in \mathbb{N}_0$ , period  $T$ , and angular frequency  $\omega_n = 2\pi n/T$ ,

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} y(t) dt \quad (1)$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(\omega_n t) dt \quad (2)$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(\omega_n t) dt. \quad (3)$$

The *Fourier synthesis* of a periodic function  $y(t)$  with analysis components  $a_n$  and  $b_n$  corresponding to  $\omega_n$  is

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t). \quad (4)$$

**3** Let's consider the complex form of the Fourier series, which is analogous to [Definition 09 four.1](#). It may be helpful to review Euler's formula(s) – see [Appendix 04.1 com.euler](#).

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<sup>1</sup>It's important to note that the symbol  $\omega_n$ , in this context, is not the natural frequency, but a frequency indexed by integer  $n$ .

**Definition 09 four.2: Fourier series: complex form**

The *Fourier analysis* of a periodic function  $y(t)$  is, for  $n \in \mathbb{N}_0$ , period  $T$ , and angular frequency  $\omega_n = 2\pi n/T$ ,

$$c_{\pm n} = \frac{1}{T} \int_{-T/2}^{T/2} y(t) e^{-j\omega_n t} dt. \quad (5)$$

The *Fourier synthesis* of a periodic function  $y(t)$  with analysis components  $c_n$  corresponding to  $\omega_n$  is

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}. \quad (6)$$

4 We call the integer  $n$  a **harmonic** and the frequency associated with it,

$$\omega_n = 2\pi n/T, \quad (7)$$

the **harmonic frequency**. There is a special name for the first harmonic ( $n = 1$ ): the **fundamental frequency**. It is called this because all other frequency components are integer multiples of it.

5 It is also possible to convert between the two representations above.

**Definition 09 four.3: Fourier series: converting between forms**

The complex Fourier analysis of a periodic function  $y(t)$  is, for  $n \in \mathbb{N}_0$  and  $a_n$  and  $b_n$  as defined above,

$$c_{\pm n} = \frac{1}{2} (a_{|n|} \mp j b_{|n|}) \quad (8)$$

The sinusoidal Fourier analysis of a periodic function  $y(t)$  is, for  $n \in \mathbb{N}_0$  and  $c_n$  as defined above,

$$a_n = c_n + c_{-n} \text{ and} \quad (9)$$

$$b_n = j(c_n - c_{-n}). \quad (10)$$

6 The **harmonic amplitude**  $C_n$  is

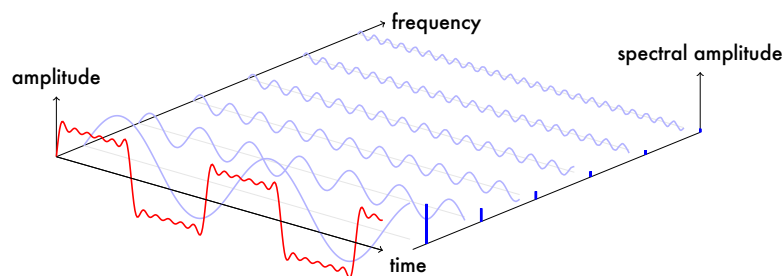
$$C_n = \sqrt{a_n^2 + b_n^2} \quad (11)$$

$$= 2\sqrt{c_n c_{-n}}. \quad (12)$$

A **magnitude line spectrum** is a graph of the harmonic amplitudes as a function of the harmonic frequencies. The **harmonic phase** is

$$\begin{aligned}\theta_n &= -\arctan_2(b_n, a_n) && \text{(see Appendix 01.2 math.trig)} \\ &= \arctan_2(\operatorname{Im}(c_n), \operatorname{Re}(c_n)). && (13)\end{aligned}$$

7 The illustration of Fig. series.1 shows how sinusoidal components sum to represent a square wave. A line spectrum is also shown.



**Figure series.1:** a partial sum of Fourier components of a square wave shown through time and frequency. The spectral amplitude shows the amplitude of the corresponding Fourier component.

8 Let us compute the associated spectral components in the following example.

### Example 09.1 four.series-1

Compute the first five harmonic amplitudes that represent the line spectrum for a square wave in the figure above.

re:  
Fourier  
series  
analysis:  
line  
spectrum

