10.1 freq.fir Frequency and impulse response

1 This lecture proceeds in three parts. First, the Fourier transform is used to derive the *frequency response function*. Second, this is used to derive the *frequency response*. Third, the frequency response for an impulse input is explored.

Frequency response functions

2 Consider a dynamic system described by the *input-output differential equation*—with variable y representing the *output*, dependent variable time t, variable u representing the *input*, constant coefficients a_i, b_j, order n, and $m \leqslant n$ for $n \in \mathbb{N}_0$—as:

$$\begin{aligned} \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = \\ b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \end{aligned} \tag{1}$$

3 The **Fourier transform** $\mathcal{F}$ of Eq. 1 yields something interesting (assuming zero initial conditions):

$$\begin{aligned} \mathcal{F}\left(\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y\right) = \\ \mathcal{F}\left(b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u\right) \Rightarrow \\ \mathcal{F}\left(\frac{d^n y}{dt^n}\right) + a_{n-1}\mathcal{F}\left(\frac{d^{n-1} y}{dt^{n-1}}\right) + \cdots + a_1 \mathcal{F}\left(\frac{dy}{dt}\right) + a_0 \mathcal{F}(y) = \\ b_m\mathcal{F}\left(\frac{d^m u}{dt^m}\right) + b_{m-1}\mathcal{F}\left(\frac{d^{m-1} u}{dt^{m-1}}\right) + \cdots + b_1 \mathcal{F}\left(\frac{du}{dt}\right) + b_0 \mathcal{F}(u) \Rightarrow \\ (j\omega)^n Y + a_{n-1}(j\omega)^{n-1} Y + \cdots + a_1 (j\omega) Y + a_0 Y = \\ b_m (j\omega)^m U + b_{m-1}(j\omega)^{m-1} U + \cdots + b_1 (j\omega) U + b_0 U. \end{aligned}$$

Solving for Y,

The inverse Fourier transform $\mathcal{F}^{-1}$ of Y is the **forced response**. However, this is not our primary concern; rather, we are interested to solve for the frequency response function $H(j\omega)$ as the ratio of the output transform Y to the input transform U, i.e.[1]

$$H(j\omega) \equiv \frac{Y(\omega)}{U(\omega)} \tag{2a}$$

$$= \frac{b_m(j\omega)^m + b_{m-1}(j\omega)^{m-1} + \cdots + b_1(j\omega) + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + a_1(j\omega) + a_0}. \tag{2b}$$

4 Note that a frequency response function can be converted to a transfer function via the substitution $j\omega \mapsto s$ and, conversely, a transfer function can be converted to a frequency response function[2] via the substitution $s \mapsto j\omega$, as in

It is often easiest to first derive a transfer function—using any of the methods described, previously—then convert this to a frequency response function.

Frequency response

5 From above, we can solve for the output response y from the frequency response function by taking the inverse Fourier transform:

$$y(t) = \mathcal{F}^{-1} Y(\omega). \tag{3}$$

[1]It is traditional to use the non-standard, single-sided Fourier transform for the frequency response function for $H(j\omega)$. The motivation is that it then pairs well with the (single-sided) Laplace transform's transfer function.

[2]A caveat is that $H(j\omega) = H(s)|_{s \mapsto j\omega}$ only holds if the corresponding single-sided Fourier transform exists.

From the definition of the frequency response function (2a),

$$y(t) = \mathcal{F}^{-1}(H(j\omega)U(\omega)). \tag{4}$$

6 The **convolution theorem** states that, for two functions of time h and u,

$$\mathcal{F}(h * u) = \mathcal{F}(h)\mathcal{F}(u) \tag{5a}$$

$$= H(j\omega)U(\omega), \tag{5b}$$

where the **convolution operator** $*$ is defined by

$$(h * u)(t) \equiv \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\, d\tau. \tag{6}$$

Therefore,

(from (5b))

(from (6))

This is the **frequency response** in terms of all time-domain functions.

Impulse response

7 The frequency response result includes an interesting object: $h(t)$. What is the physical significance of h, other than its definition, as the inverse Fourier transform of $H(j\omega)$?

8 Consider the singularity input $u(t) = \delta(t)$, an impulse. The frequency response is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)\, d\tau. \tag{7}$$

The so-called **sifting property** of δ yields

$$y(t) = h(t). \tag{8}$$

That is, h is the **impulse response**.

9 A very interesting aspect of this result is that

$$H(j\omega) = \mathcal{F}(h). \tag{9}$$

That is, the Fourier transform of the impulse response is the frequency response function. A way to estimate, via measurement, the frequency response function (and transfer function) of a system is to input an impulse, measure and fit the response, then Fourier transform it. Of course, putting in an actual impulse and fitting the response, perfectly are impossible; however, estimates using approximations remain useful.

10 It is worth noting that frequency response/transfer function estimation is a significant topic of study, and many techniques exist. Another method is described in Lec. 10.2 freq.sin.

Example 10.1 freq.fir-1

re: impulse response estimation of $H(j\omega)$

Estimate the frequency response function $H(j\omega)$ of a system from impulse response $h(t)$ "data". (We'll generate this data ourselves, simulating a measured impulse response.) We will not attempt to find the functional form of $H(j\omega)$, just its "numerical" form, i.e. we'll plot our estimate of the spectrum.

Note that if we wanted to find a functional estimate of $H(j\omega)$, it would behoove us to use Matlab's System Identification Toolbox.

Generate impulse response data

We need a system to simulate to get this (supposedly "measured") data. Let's define a transfer function

$$H(s) = \frac{s + 20}{s^2 + 4s + 20}. \tag{10}$$

```
sys = tf([1,20],[1,4,20])
```

```
sys =

      s + 20
  --------------
```

```
  s^2 + 4 s + 20

Continuous-time transfer function.
```

What are the poles?

```
poles = pole(sys)
```

```
poles =

  -2.0000 + 4.0000i
  -2.0000 - 4.0000i
```

This corresponds to a damped oscillator with natural frequency as follows.

```
abs(poles(1))
```

```
ans =

    4.4721
```

Now let's find the impulse response.

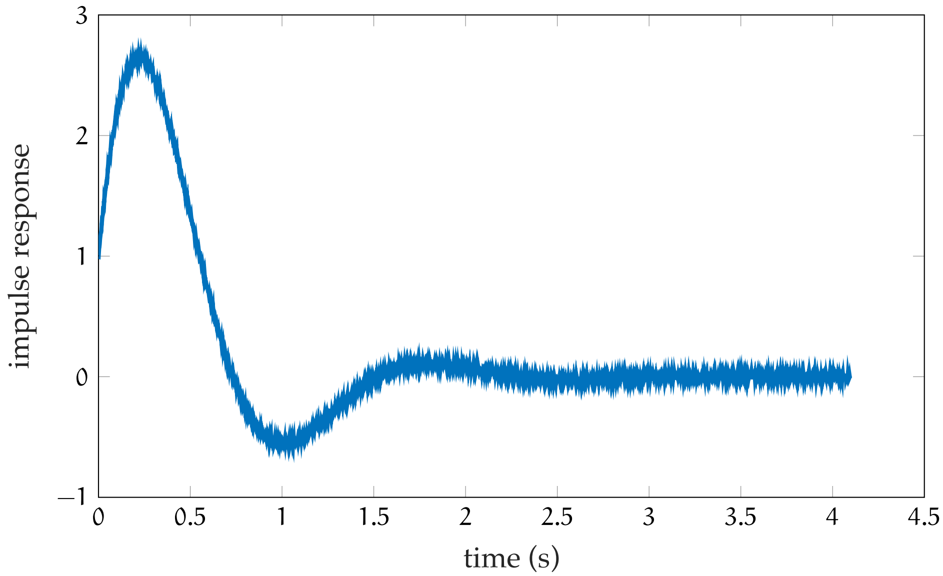
```
fs = 1000; % Hz .. sampling frequency
N = 2^12;
t_a = 0:1/fs:(N-1)/fs;
h_a = impulse(sys,t_a);
```

To make this seem a little more realistic as a "measurement," we should add some noise.

```
noise = 0.01*randn(N,1);
h_noisy = h_a + noise;
```

Plot the impulse response.

```
figure
plot(...
  t_a,h_noisy, ...
  'linewidth',1.5 ...
)
xlabel('time (s)')
ylabel('impulse response')
```



Discrete Fourier transform

The discrete Fourier transform will give us an estimate of the frequency spectrum of the system; that is, a numerical version of $H(j\omega)$.

```
H = fft(h_noisy);
```

Compute the one-sided magnitude spectrum.

```
H_mag = abs(H/fs); % note the scaling
H_mag = H_mag(1:N/2+1); % first half, only
```

Compute the one-sided phase spectrum.

```
H_pha = angle(H); % note the scaling
H_pha = H_pha(1:N/2+1); % first half, only
```

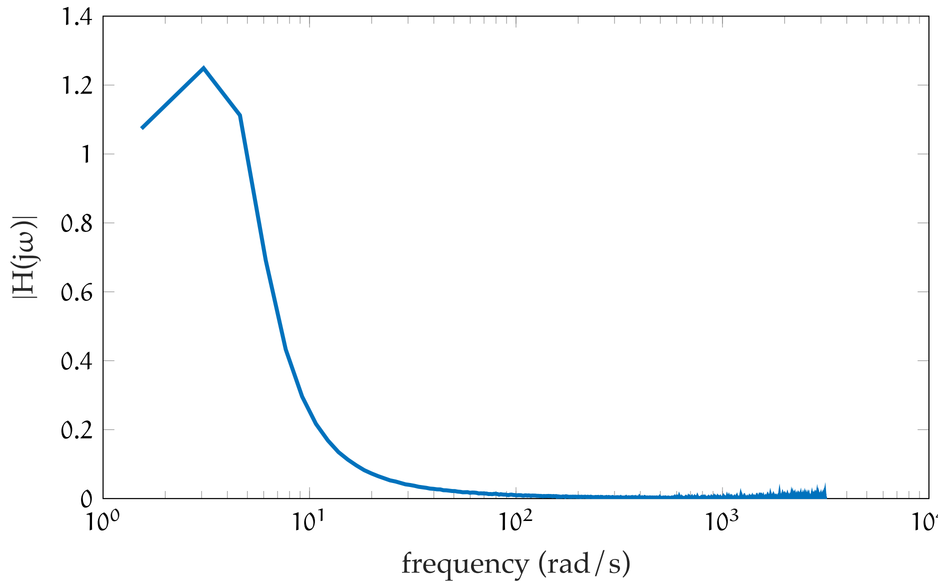
Now the corresponding frequencies.

```
f = fs*(0:(N/2))/N;
```

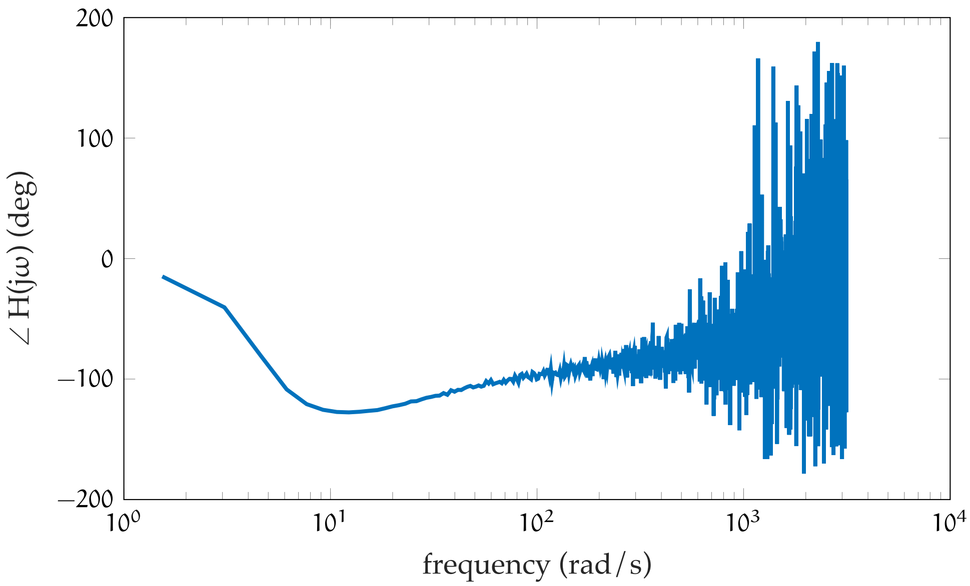
Plot the frequency response function

We like to use a logarithmic scale, at least in frequency, for the spectrum plots.

```
figure
semilogx(...
  2*pi*f,H_mag, ...
  'linewidth',1.5 ...
)
xlabel('frequency (rad/s)')
ylabel('|H(j\omega)|')
```



```
figure
semilogx(...
  2*pi*f,180/pi*H_pha, ...
  'linewidth',1.5 ...
)
xlabel('frequency (rad/s)')
ylabel('\angle H(j\omega) (deg)')
```



When the magnitude $|H(j\omega)|$ is small, the signal-to-noise ratio is so low that the phase estimates are dismal. This can be mitigated by increasing sample-size and using more advanced techniques for estimating $H(j\omega)$, such as those available in Matlab's System Identification Toolbox.