10.2 freq.sin Sinusoidal input, frequency response

1 In this lecture, we explore the relationship—which turns out to be pretty chummy—between a system's frequency response function $H(j\omega)$ and its sinusoidal forced response.

2 Let's build from the frequency response function $H(j\omega)$ definition:

$$y(t) = \mathcal{F}^{-1}Y(\omega) \tag{1a}$$
$$= \mathcal{F}^{-1}(H(j\omega)U(\omega)). \tag{1b}$$

We take the input to be sinusoidal, with amplitude $A \in \mathbb{R}$, angular frequency ω_0, and phase ψ:

$$u(t) = A\cos(\omega_0 t + \psi). \tag{2}$$

The Fourier transform of the input, $U(\omega)$, can be constructed via transform identities from Table ft.1. This takes a little finagling. Let

$$p(t) = Aq(t), \tag{3a}$$
$$q(t) = r(t - t_0), \text{ and} \tag{3b}$$
$$r(t) = \cos\omega_0 t, \text{ where} \tag{3c}$$
$$t_0 = -\psi/\omega_0. \tag{3d}$$

The corresponding Fourier transforms, from Table ft.1, are

$$P(\omega) = AQ(\omega), \tag{4a}$$
$$Q(\omega) = e^{-j\omega t_0}R(\omega), \text{ and} \tag{4b}$$
$$R(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0). \tag{4c}$$

Putting these together,

(because δs)

3 And now we are ready to substitute into Eq. 1b; also applying the "linearity" property of the Fourier transform:

$$y(t) = A\pi \left(e^{j\psi} \mathcal{F}^{-1}(H(j\omega)\delta(\omega - \omega_0)) + e^{-j\psi} \mathcal{F}^{-1}(H(j\omega)\delta(\omega + \omega_0)) \right). \tag{5}$$

The definition of the inverse Fourier transform gives

$$\begin{aligned} y(t) = \frac{A}{2} \Bigg(& e^{j\psi} \int_{-\infty}^{\infty} e^{j\omega t} H(j\omega)\delta(\omega - \omega_0) d\omega + \\ & + e^{-j\psi} \int_{-\infty}^{\infty} e^{j\omega t} H(j\omega)\delta(\omega + \omega_0) d\omega \Bigg). \end{aligned} \tag{6}$$

Recognizing that δ is an even distribution ($\delta(t) = \delta(-t)$) and applying the sifting property of δ allows us to evaluate each integral:

$$y(t) = \frac{A}{2} \left(e^{j\psi} e^{j\omega_0 t} H(j\omega_0) + e^{-j\psi} e^{-j\omega_0 t} H(-j\omega_0) \right). \tag{7}$$

Writing H in polar form,

(8)

The Fourier transform is conjugate symmetric—that is, $F(-\omega) = F^*(\omega)$—which allows us to further simply:

$$y(t) = \frac{A|H(j\omega_0)|}{2} \left(e^{j(\omega_0 t + \psi)} e^{j\angle H(j\omega_0)} + e^{-j(\omega_0 t + \psi)} e^{-j\angle H(j\omega_0)} \right) \tag{9a}$$

$$= A|H(j\omega_0)| \frac{e^{j(\omega_0 t + \psi + \angle H(j\omega_0))} + e^{-j(\omega_0 t + \psi + \angle H(j\omega_0))}}{2}. \tag{9b}$$

Finally, Euler's formula yields something that deserves a box.

Equation 10 sinusoidal response in terms of $H(j\omega)$

4 This is a remarkable result. For an input sinusoid, a linear system has a forced response that

- is also a sinusoid,
- is at the same frequency as the input,
- differs only in amplitude and phase,
- differs in amplitude by a factor of $|H(j\omega)|$, and
- differs in phase by a shift of $\angle H(j\omega)$.

Now we see one of the key facets of the frequency response function: it governs how a sinusoid transforms through a system. And just think how powerful it will be once we combine it with the powerful principle of superposition and the mighty Fourier series representation of a function—as a "superposition" of sinusoids!