11.3 lap.pr Properties of the Laplace transform

1 The Laplace transform has several important properties, several of which follow from the simple fact of its ______________ definition. We state the properties without proof, but several are easy to show and make good exercises.

Existence

2 As we have already seen, the Laplace transform exists for more functions than does the Fourier transform. Let $f : \mathbb{R}_+ \to \mathbb{R}$ have a finite number of finite-magnitude discontinuites. If there can be found $M, \alpha \in \mathbb{R}$ such that

$$|f(t)| \leqslant Me^{\alpha t} \quad \forall t \in \mathbb{R}_+ \tag{1}$$

then the transform exists (converges) for $\sigma > \alpha$.

3 Note that this is a ______________ condition, not necessary. That is, there may be (and are) functions for which a transform exists that do not meet the condition above.

Linearity

4 The Laplace transform is a ______________ map. Let $a, b \in \mathbb{R}$; $f, g \in T$ where T is a set of functions of nonnegative time t; and F, G the Laplace transform images of f, g. The following identity holds:

$$\mathcal{L}(af(t) + bg(t)) = aF(s) + bG(s). \tag{2}$$

Time-shifting

5 Shifting the time-domain function $f(t)$ in time corresponds to a simple product in the s-domain Laplace transform image. Let the Laplace transform image of $f(t)$ be $F(s)$ and $\tau \in \mathbb{R}$. The following identity holds:

$$\mathcal{L}(f(t + \tau)) = e^{s\tau}F(s). \tag{3}$$

Time-differentiation

6 ____________________ the time-domain function $f(t)$ with respect to time yields a simple relation in the s-domain. Let $F(s)$ be the Laplace transform image of $f(t)$ and $f(0)$ the value of f at $t = 0$. The following identity holds:[7]

$$\mathcal{L}\frac{df}{dt} = sF(s) - f(0). \tag{4}$$

Time-integration

7 Similarly, _________________ the time-domain function $f(t)$ with respect to time yields a simple relation in the s-domain. Let $F(s)$ be the Laplace transform image of $f(t)$. The following identity holds:[8]

$$\mathcal{L}\int_0^t f(\tau)\, d\tau = \frac{1}{s}F(s). \tag{5}$$

Convolution

8 The convolution operator $*$ is defined for real functions of time f, g by

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\, d\tau. \tag{6}$$

This too has a simple Laplace transform. Let F, G be the Laplace transforms of f, g. The following identity holds:

$$\mathcal{L}(f * g)(t) = F(s)G(s). \tag{7}$$

Final value theorem

9 The **final value theorem** is a property of the Laplace transform. This theorem allows the computation of _______________ time-domain steady-state values from the frequency domain, which can be quite convenient when the inverse Laplace transform is elusive. Let $f(t)$ have

[7] For this reason, it is common for s to be called the *differentiator*, but this is imprecise and pretty bush league.

[8] For this reason, it is common for $1/s$ to be called the *integrator*.

tranform $F(s)$ and its time-derivative have an existing transform. If the limit in time exists,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s). \tag{8}$$

Note that if the steady-state of $f(t)$ is not a constant (e.g. it is sinusoidal), the limit does not exist.