14.1 nlin.lin Linearization

1 A common method for dealing with a nonlinear system is to **linearize** it: transform it such that its state equation is linear. A linearized model is typically only valid in some neighborhood of state-space. This neighborhood is selected by choosing an **operating point** x_0 used in the linearization process. We use two considerations when choosing an operating point:

- 1. that implied by the name—it should be in a region of state-space in which the state will stay throughout the system's operation—and
- 2. the validity of the model near the operating point.

Due to the fact that nonlinear systems tend to be more-linear near equilibria, the second consideration frequently suggests we choose one as an operating point: $\mathbf{x}_{o} = \overline{\mathbf{x}}$.

Taylor series expansion

2 A Taylor series expansion of Eq. 1a about an operating point x_0 , u_0 (for a nonautonomous system) yields polynomial terms that are linear, quadratic, etc. in x and u. If we keep only the linear terms and define new state and input variables

$$\mathbf{x}^* = \mathbf{x} - \mathbf{x}_o$$
 and $\mathbf{u}^* = \mathbf{u} - \mathbf{u}_o$ (1a)

we get a linear state equation

$$\frac{\mathrm{d}\mathbf{x}^*}{\mathrm{d}\mathbf{t}} = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \tag{1b}$$

where the matrix components are given by

$$A_{ij} = \frac{\partial f_i}{\partial x_j}\Big|_{\mathbf{x}_o, \mathbf{u}_o} \qquad \text{and} \qquad B_{ij} = \frac{\partial f_i}{\partial u_j}\Big|_{\mathbf{x}_o, \mathbf{u}_o}. \tag{1c}$$

These first-derivative matrices are generally called Jacobian matrices.This result also applies to autonomous equations if we drop the Bu* term.

Example 14.1 nlin.lin-1

Consider a vehicle suspension system that is overloaded such that its springs are exhibiting *hardening* behavior such that a lumped-parameter constitutive equation for the springs (collectively) is

$$f_k = kx_k + ax_k^3 \tag{2}$$

where f_k is the force, x_k the displacement, and k, a > 0 constant parameters of the spring.

- a. Develop a (nonlinear) spring-mass-damper linear graph model for the vehicle suspension with input position source X_s.
- b. Derive a nonlinear state-space model from the linear graph model using the state vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_{\mathrm{m}} & \mathbf{v}_{\mathrm{m}} \end{bmatrix}^{\top} . \tag{3}$$

c. Linearize the system about the operating point

$$\mathbf{x}_{o} = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\top}$$
 and $\mathbf{u}_{o} = \begin{bmatrix} 0 \end{bmatrix}$ (4)

by computing the A, B, and E matrices of the linearized system.^a

re: hardening spring

^{*a*}The E matrix is the Jacobian with respect to the time-derivative of the input: \dot{u} , which arises occasionally.