02.1 adv.eig Systems with repeated eigenvalues

This topic is fully treated by Brogan (1991, p 250), but not by Rowell **and** Wormley (1997). Every $n \times n$ matrix has n eigenvalues, and for each distinct eigenvalue λ_i, a linear independent eigenvector $\mathbf{m}_i$ exists. For every eigenvalue λ_i repeated μ_i times (termed *algebraic multiplicity* of λ_i), any number q_i (termed *geometric multiplicity* or *degeneracy* of λ_i) up to and including μ_i of independent eigenvectors may exist: $1 \leqslant q_i \leqslant \mu_i$. q_i is equal to the dimension of the null space of $A - \mathbf{I}\lambda_i$,

$$q_i = n - \operatorname{rank}(A - \lambda_i \mathbf{I}). \tag{1}$$

This gives rise to the three cases that follow.

full degeneracy When $q_i = \mu_i$, the eigenvalue problem has $q_i = \mu_i$ independent solutions for $\mathbf{m}_i$. So, even though there were not n distinct eigenvalues, n distinct eigenvectors still exist and we can diagonalize or decouple the system as before.

simple degeneracy When $q_i = 1$, the eigenvalue problem has $q_i = 1$ independent solutions for $\mathbf{m}_i$. We would still like to construct a basis set of n independent vectors, but they can no longer be eigenvectors, and we will no longer be able to fully diagonalize or decouple the system. There are multiple ways of doing this (e.g. Gram-Schmidt), but the typical and most nearly diagonal way is to construct $\mu_i - q_i$ *generalized eigenvectors* (here also called $\mathbf{m}_i$), which will be included in the modal matrix M along with the eigenvectors. The generalized eigenvectors are found by solving the usual eigenvalue/vector problem for the first eigenvector $\mathbf{m}_i^1$ corresponding to λ_i, then solving it again with the following equations to find the generalized eigenvectors

$$\begin{aligned}
(A - \lambda_i)\mathbf{m}_i^2 &= \mathbf{m}_i^1 \\
(A - \lambda_i)\mathbf{m}_i^3 &= \mathbf{m}_i^2 \\
&\vdots
\end{aligned}$$

This forms the modal matrix M. The block-diagonal *Jordan form* matrix J, analogous to the diagonal $\mathbf{\Lambda}$ is

$$J = M^{-1}AM, \tag{2}$$

which gives the most-decoupled state transition matrix

$$\Phi(t) = Me^{Jt}M^{-1}. \tag{3}$$

general degeneracy If $1 < q_i < \mu_i$, the preceding method applies, but it may be ambiguous as to which eigenvector the generalized eigenvectors correspond (or how many for each). This can be approached by trial and error or a systematic method presented by Brogan (1991, p 255).

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