4. There is one node not connected to the voltage source, for which the KCL equation is
$$i_C = i_R.$$
5. There is one loop which has KVL equation is
$$v_R + v_C - V_s = 0.$$
6. Solve.
   1. Eliminate i_C and v_R using KCL and KVL to yield the following.

$$\begin{array}{ll} C & \dfrac{dv_C}{dt} = \dfrac{1}{C} i_R \\ R & \quad i_R = \dfrac{1}{R}(V_s - v_C) \end{array}$$

   2. Substituting the R equation into the C equation, we eliminate i_R:
$$\frac{dv_C}{dt} = \frac{1}{RC}(V_s - v_C).$$
   3. Rearranging,
$$RC\frac{dv_C}{dt} + v_C = V_s.$$
   Let's solve the ODE with initial condition $v_C(0) = 0$.
      1. Homogeneous solution. The characterestic equation is
$$RC\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{RC}.$$
      Therefore, $v_{Ch}(t) = k_1 e^{-t/(RC)}$.
      2. Particular solution. The form of the input $V_s(t)$ invites us to assume $v_{Cp}(t) = k_2$. Substituting into the ODE,
$$RC(0) + k_2 = 12 \text{ V} \quad \Rightarrow \quad k_2 = 12 \text{ V}.$$
      3. Total solution: $v_C(t) = v_{Ch}(t) + v_{Cp}(t)$.
      4. Specific solution:
$$v_C(0) = 0 \quad \Rightarrow \quad k_1 + k_2 = 0 \quad \Rightarrow \quad k_1 = -k_2 = -12\text{V}.$$
      Since, $v_o(t) = v_C(t)$,
$$v_o(t) = 12(1 - e^{-t/(RC)}).$$

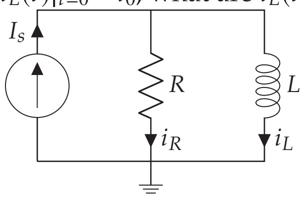
2.3 A Sinusoidal Input Example

Notice that we have yet to talk about **alternating current (ac) circuit analysis** or **direct current (dc) circuit analysis**. In fact, these ambiguous terms can mean a few different things. Approximately, an ac circuit analysis is one for which the input is sinusoidal and a dc circuit analysis is one for which the input is a constant. This ignores **transient response** (early response when the initial-condition response dominates) versus **steady-state response** (later response when the initial-condition response has decayed) considerations. We'll consider this more in (**lec:transient_steady**).

We have remained general enough to be able to handle sinusoidal and constant sources in both transient and steady-state response. (**ex:rc_circuit_analysis_01**) features a circuit with a constant voltage source and a capacitor. Now we consider circuit with a sinusoidal current source and an inductor because why change only one thing when you could change more?

Example 2.2

Given the RL circuit shown, current input $I_s(t) = A \sin \omega t$, and initial condition $i_L(t)|_{t=0} = i_0$, what are $i_L(t)$ and $v_L(t)$ for $t \geq 0$?.



1. The circuit diagram is given.
2. The signs are given.
3. The n elemental equations are as follows.

$$L \quad \frac{di_L}{dt} = \frac{1}{L} v_L \tag{2.1}$$

$$R \quad v_R = i_R R \tag{2.2}$$

4. There are actually two nodes not connected to the voltage source, but they give the same KCL equation

$$i_R = I_s - i_L.$$

5. There is one loop that doesn't have a current source in it, for which the KVL equation is

$$v_L = v_R.$$

6. Solve.
    1. Eliminate v_L and i_R using KCL and KVL to yield the following.

$$L \quad \frac{di_L}{dt} = \frac{1}{L} v_R \tag{2.3}$$

$$R \quad v_R = (I_s - i_L)R \tag{2.4}$$

2. Substituting the R equation into the L equation, we eliminate v_R to obtain

$$\frac{di_L}{dt} = \frac{R}{L}(I_s - i_L).$$

3. Rearranging and letting $\tau = L/R$,

$$\tau \cdot \frac{di_L}{dt} + i_L = I_s.$$

Let's solve the ODE with initial condition $i_L(0) = i_0$.

1. Homogeneous solution. The characterestic equation is

$$\tau\lambda + 1 = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{\tau}.$$

   Therefore, $i_{Lh}(t) = \kappa_1 e^{-t/\tau}$.

2. Particular solution. The form of the input $I_s(t)$ invites us to assume $i_{Lp}(t) = k_1 \sin \omega t + k_2 \cos \omega t$. Substituting into the ODE,

$$\begin{aligned}
&\tau\omega(k_1 \cos \omega t - k_2 \sin \omega t) + \\
&\qquad (k_1 \sin \omega t + k_2 \cos \omega t) = A \sin \omega t \quad \Rightarrow \\
&\tau\omega k_1 + k_2 = 0 \quad \text{and} \quad -\tau\omega k_2 + k_1 = A \quad \Rightarrow \\
&k_2 = -\tau\omega k_1 \quad \text{and} \quad (\tau\omega)^2 k_1 + k_1 = A \quad \Rightarrow \\
&k_1 = \frac{A}{(\tau\omega)^2 + 1} \quad \text{and} \quad k_2 = -\frac{A\tau\omega}{(\tau\omega)^2 + 1}
\end{aligned}$$

   Success! So we have

$$i_{Lp}(t) = \frac{A}{(\tau\omega)^2 + 1}(\sin \omega t - \tau\omega \cos \omega t).$$

3. Total solution:

$$\begin{aligned}
i_L(t) &= i_{Lh}(t) + i_{Lp}(t) \\
&= \kappa_1 e^{-t/\tau} + \frac{A}{(\tau\omega)^2 + 1}(\sin \omega t - \tau\omega \cos \omega t).
\end{aligned}$$

4. Specific solution:

$$
\begin{aligned}
i_L(0) = i_0 \quad &\Rightarrow \\
\kappa_1 - \frac{A\tau\omega}{(\tau\omega)^2+1} = i_0 \quad &\Rightarrow \\
\kappa_1 = i_0 + \frac{A\tau\omega}{(\tau\omega)^2+1}&.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
i_L(t) = \left(i_0 + \frac{A\tau\omega}{(\tau\omega)^2+1}\right) e^{-t/\tau} &+ \\
\frac{A}{(\tau\omega)^2+1}\left(\sin\omega t - \tau\omega\cos\omega t\right)&.
\end{aligned}
$$

Now, a better way to write this is as a single sinusoid. The two-to-one formulas can be applied to obtain

$$
\begin{aligned}
i_L(t) = \left(i_0 + \frac{A\tau\omega}{(\tau\omega)^2+1}\right) e^{-t/\tau} &+ \\
\frac{A}{\sqrt{(\tau\omega)^2+1}} \sin(\omega t - \arctan\tau\omega)&.
\end{aligned}
$$

We can see clearly that there is a sinusoidal component and a decaying exponential component.

7. Finally, we can find $v_L(t)$ from the inductor elemental equation:

$$
\begin{aligned}
v_L(t) &= L\frac{di_L}{dt} \\
&= -\frac{L}{\tau}\left(i_0 + \frac{A\tau\omega}{(\tau\omega)^2+1}\right) e^{-t/\tau} + \\
&\quad \frac{AL\omega}{\sqrt{(\tau\omega)^2+1}} \cos(\omega t - \arctan\tau\omega).
\end{aligned}
$$

8. See section 2.4 for plots.

2.4 Transient and Steady-State Response

Let's consider them response of the circuit in example 2.2. We found that the inductor had current and voltage responses

$$i_L(t) = \left(i_0 + \frac{A\tau\omega}{(\tau\omega)^2+1}\right)e^{-t/\tau} + \frac{A}{\sqrt{(\tau\omega)^2+1}}\sin(\omega t - \arctan\tau\omega)$$

and

$$v_L(t) = -\frac{L}{\tau}\left(i_0 + \frac{A\tau\omega}{(\tau\omega)^2+1}\right)e^{-t/\tau} + \frac{AL\omega}{\sqrt{(\tau\omega)^2+1}}\cos(\omega t - \arctan\tau\omega).$$

Note that the top line of each of these equations decays exponentially to zero. The response while this term dominates is the *transient response* and the response thereafter is the *steady-state response*.

In 6τ (six time constants) the exponential term has decayed to less than 1 %, so we often assume the other term will be dominating by that point.

We will plot $i_L(t)$ and $v_L(t)$ from above to illustrate transient and steady-state response.

Let's use Python to plot the response. We begin by loading two packages as follows:

```python
import numpy as np  # For numerical calculations
import matplotlib.pyplot as plt  # For plotting
```

Plots cannot be created without some definition of parameters. Let us define them as follows:

```python
R = 1  # (Ohms) resistance
L = 1e-3  # (H) inductance
i_0 = 10  # (A) initial current in inductor
A = 10  # (A) sinusoidal input amplitude
omega = 5e3  # (rad/s) sinusoidal input angular frequency
tau = L/R  # (s) time constant
```

The current and voltage can be defined as follows: