3.2 Impedance

With complex representations for voltage and current, we can introduce the concept of **impedance**.

Definition 3.1

Impedance Z is the complex ratio of voltage v to current i of a circuit element:

$$Z = \frac{v}{i}.$$

The real part $\Re(Z)$ is called the **resistance** and the imaginary part $\Im(Z)$ is called the **reactance**. As with complex voltage and current, we can represent the impedance as a *phasor*.

Note that definition 3.1 is a generalization of Ohm's law. In fact, we call the following expression **generalized Ohm's law**:

$$v = iZ.$$

3.2.1 Impedance of Circuit Elements

The impedance of each of the three passive circuit elements we've considered thus far are listed, below. Wherever it appears, ω is the angular frequency of the element's voltage and current.

Resistor For a resistor with resistance R, the impedance is all real:

$$Z_R = Re^{j0} = R.$$

Capacitor For a capacitor with capacitance C, the impedance is all imaginary:

$$Z_C = \frac{1}{\omega C} e^{-j\pi/2} = \frac{1}{j\omega C}.$$

Inductor For an inductor with inductance L, the impedance is all imaginary:

$$Z_L = \omega L e^{j\pi/2} = j\omega L.$$

These are represented in the complex plane in figure 3.4.

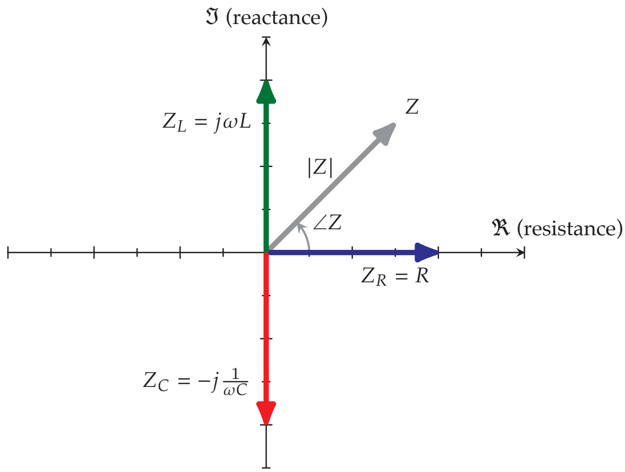


Figure 3.4. The impedances of a resistor Z_R, a capacitor Z_C, and an inductor Z_L in the complex plane.

3.2.2 Combining the Impedance of Multiple Elements

As with resistance, the impedance of multiple elements may be combined to find an **effective impedance**.

K elements with impedances Z_j connected in *series* have equivalent impedance Z_e given by the expression

$$Z_e = \sum_{j=1}^{K} Z_j.$$

K elements with impedances Z_j connected in *parallel* have equivalent impedance Z_e given by the expression

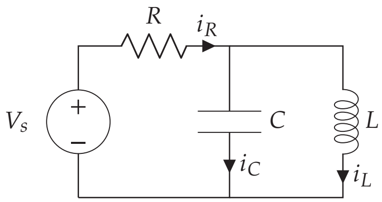
$$Z_e = 1 \left/ \sum_{j=1}^{K} 1/Z_j \right. .$$

In the special case of two elements with impedances Z_1 and Z_2,

$$\begin{aligned} Z_e &= \frac{1}{1/Z_1 + 1/Z_2} \\ &= \frac{Z_1 Z_2}{Z_1 + Z_2}. \end{aligned}$$

Example 3.1

Given the circuit shown with voltage source $V_s(t) = Ae^{j\phi}$, what is the total impedance at the source?



The total impedance is the combination of the series resistor impedance with the parallel capacitor and inductor impedances:

$$\begin{aligned}
Z_e &= Z_R + \frac{Z_L Z_C}{Z_L + Z_C} \\
&= R + \left(\omega L e^{j\pi/2} \frac{1}{\omega C} e^{-j\pi/2}\right) \bigg/ \left(j\omega L - j\frac{1}{\omega C}\right) \\
&= R + \frac{L/C}{j(LC\omega^2 - 1)/(\omega C)} \\
&= R + \frac{L\omega}{j(LC\omega^2 - 1)} \\
&= R - j\frac{L\omega}{LC\omega^2 - 1} \\
&= A' e^{j\phi'}
\end{aligned}$$

where we can compute the magnitude A' and phase ϕ' to be

$$A' = \sqrt{R^2 + \left(-\frac{L\omega}{LC\omega^2 - 1}\right)^2}$$

$$\phi' = \arctan\left(-\frac{L\omega}{LC\omega^2 - 1} \bigg/ R\right).$$

Note that we used the fact that it's easier to multiply and divide phasors and add and subtract rectangular representations.

3.3 Methodology for Impedance-Based Circuit Analysis

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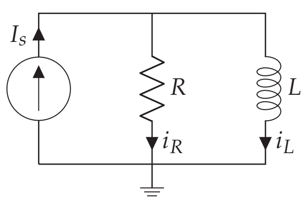
It turns out we can follow essentially the same algorithm presented in section 2.2 for analyzing circuits in steady-state with impedance. There are enough variations that we re-present it here.

Let n be the number of passive circuit elements in a circuit, which gives $2n$ (v and i for each element) unknowns. The method is this.

1. Draw a *circuit diagram*.
2. Label the circuit diagram with the *sign convention* by labeling each element with the "assumed" direction of current flow.
3. Write *generalized Ohm's law* for each circuit element and define the impedance of each element.
4. For every node not connected to a voltage source, write Kirchhoff's current law (KCL).
5. For each loop not containing a current source, write Kirchhoff's voltage law (KVL).
6. You probably have a linear system of $2n$ *algebraic* equations (and $2n$ unknowns) to be solved simultaneously. If only certain variables are of interest, these can be found by eliminating other variables such that the remaining system is smaller. The following steps can facilitate this process.
    1. Eliminate n (half) of the unknowns by substitution into the elemental equations (generalized Ohm's law equations).
    2. Try substition to eliminate to get down to only those variables of interest and inputs.
    3. Solve the remaining system of linear algebraic equations for the unknowns of interest.

Example 3.2

Given the RL circuit shown with current input $I_s(t) = A \sin \omega t$, what are $i_L(t)$ and $v_L(t)$ in steady-state?. Note that this is very similar to example 2.2, but we will use impedance methods.



1. The circuit diagram is given.
2. The signs are given.