3.3 Methodology for Impedance-Based Circuit Analysis

G7

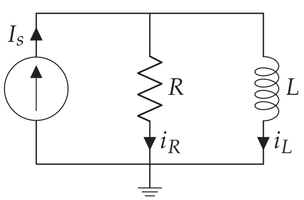
It turns out we can follow essentially the same algorithm presented in section 2.2 for analyzing circuits in steady-state with impedance. There are enough variations that we re-present it here.

Let n be the number of passive circuit elements in a circuit, which gives $2n$ (v and i for each element) unknowns. The method is this.

1. Draw a *circuit diagram*.
2. Label the circuit diagram with the *sign convention* by labeling each element with the "assumed" direction of current flow.
3. Write *generalized Ohm's law* for each circuit element and define the impedance of each element.
4. For every node not connected to a voltage source, write Kirchhoff's current law (KCL).
5. For each loop not containing a current source, write Kirchhoff's voltage law (KVL).
6. You probably have a linear system of $2n$ *algebraic* equations (and $2n$ unknowns) to be solved simultaneously. If only certain variables are of interest, these can be found by eliminating other variables such that the remaining system is smaller. The following steps can facilitate this process.
    1. Eliminate n (half) of the unknowns by substitution into the elemental equations (generalized Ohm's law equations).
    2. Try substition to eliminate to get down to only those variables of interest and inputs.
    3. Solve the remaining system of linear algebraic equations for the unknowns of interest.

Example 3.2

Given the RL circuit shown with current input $I_s(t) = A \sin \omega t$, what are $i_L(t)$ and $v_L(t)$ in steady-state?. Note that this is very similar to example 2.2, but we will use impedance methods.



1. The circuit diagram is given.
2. The signs are given.

3. The n elemental equations are as follows

$$L \quad v_L = i_L Z_L \tag{3.1}$$

$$R \quad v_R = i_R Z_R \tag{3.2}$$

   where $Z_L = j\omega L$ and $Z_R = R$.
4. There are actually two nodes not connected to the voltage source, but they give the same KCL equation

$$i_R = I_s - i_L.$$

5. There is one loop that doesn't have a current source in it, for which the KVL equation is

$$v_L = v_R.$$

6. Solve.
   1. Eliminate v_L and i_R using KCL and KVL to yield the following.

$$L \quad i_L = v_R / Z_L \tag{3.3}$$

$$R \quad v_R = (I_s - i_L) Z_R \tag{3.4}$$

   2. Substituting the R equation into the L equation, we eliminate v_R to obtain

$$i_L = \frac{Z_R}{Z_L}(I_s - i_L).$$

   3. Solving,

$$i_L = \frac{Z_R}{Z_R + Z_L} I_s.$$

All that remains is to substitute, noting that we're using the cosine form of the phasor,

$$\begin{aligned}
i_L &= \frac{R}{R + j\omega L} A e^{-j\pi/2} \\
&= \frac{R e^{j0}}{M_1 e^{j\phi_1}} A e^{-j\pi/2} \\
&= \frac{R}{M_1} A e^{-j\phi_1} e^{-j\pi/2} \\
&= \frac{R}{M_1} A e^{-j(\phi_1 + \pi/2)} \\
&= \frac{R}{M_1} A \cos(\omega t - \phi_1 - \pi/2) \\
&= \frac{R}{M_1} A \sin(\omega t - \phi_1)
\end{aligned}$$

where

$$M_1 = \sqrt{R^2 + (L\omega)^2} \text{ and}$$

$$\phi_1 = \arctan(L\omega/R).$$

Note that we could have used the unconventional sine form of the phasor. Also note that (i) this was somewhat easier than how we did it in example 2.2 and (ii) the result simply scales the amplitude and shifts the phase, as expected.

7. Finally, we can find $v_L(t)$ from the inductor elemental equation (generalized Ohm's law):

$$\begin{aligned}
v_L(t) &= i_L Z_L \\
&= \omega L e^{j\pi/2} \frac{R}{M_1} A e^{-j(\phi_1 + \pi/2)} \\
&= \frac{RL\omega}{M_1} A e^{-j\phi_1} \\
&= \frac{RL\omega}{M_1} A \cos(\omega t - \phi_1).
\end{aligned}$$

3.4 Voltage and Current Dividers

In section 1.2, we developed the useful voltage divider formula for quickly analyzing how voltage divides among series resistors. This can be considered a special case of a more general voltage divider equation for any elements described by an impedance. After developing the voltage divider, we also introduce the current divider, which divides an input current among parallel elements.

3.4.1 Voltage Dividers

First, we develop the solution for the two-element voltage divider shown in figure 3.5. We choose the voltage across Z_2 as the output. The analysis can follow our usual methodology of six steps, solving for v_2.

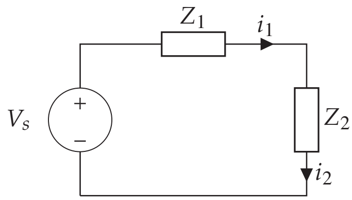


Figure 3.5. The two-element voltage divider.

1. The circuit diagram is given in figure 3.5.
2. The assumed directions of positive current flow are given in figure 3.5.
3. The elemental equations are just generalized Ohm's law equations.
$$\begin{array}{c|l} Z_1 & v_1 = i_1 Z_1 \\ Z_2 & v_2 = i_2 Z_2 \end{array}$$
4. The KCL equation is $i_2 = i_1$.
5. The KVL equation is $v_1 = V_s - v_2$.
6. Solve.
    1. Eliminating i_2 and v_1 from KCL and KVL, our elemental equations become the following.
    $$\begin{array}{c|l} Z_1 & i_1 = v_1/Z_1 = (V_s - v_2)/Z_1 \\ Z_2 & v_2 = i_1 Z_2 \end{array}$$
    2. Eliminating i_1,
    $$v_2 = \frac{Z_2}{Z_1}(V_s - v_2).$$