Vector space (Based on Marsden + Hoffman's Real Analysis)

Linear algebra revolves around a mathematical structure called a vector space. Vector spaces define what a vector is.

The components of our vectors are typically real numbers, so they "live" in real vector spaces. These are defined as follows.

Definition A real vector space V is a set of elements that we call vectors, with given operations of
vector addition $+ : V \times V \longrightarrow V$ (i.e. for $\vec{v}, \vec{w} \in V$, $(\vec{v}, \vec{w}) \mapsto \vec{v} + \vec{w} \in V$) +
scalar multiplication $\cdot : \mathbb{R} \times V \longrightarrow V$ (i.e for $a \in \mathbb{R}$ + $\vec{v} \in V$, $(a, \vec{v}) \mapsto a\vec{v} \in V$),
such that:

1. (commutativity) $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for every $\vec{v}, \vec{w} \in V$.

2. (associativity) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$ for every $\vec{v}, \vec{w}, \vec{u} \in V$.

3. (zero vector) There is a zero vector $\vec{0}$ such that $\vec{v} + \vec{0} = \vec{v}$ for every $\vec{v} \in V$.

4. (negatives) For each $\vec{v} \in V$, there is a vector $-\vec{v} \in V$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.

5. (distributivity) For $\lambda \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$, $\lambda \cdot (\vec{v} + \vec{w}) = \lambda \cdot \vec{v} + \lambda \cdot \vec{w}$.

6. (associativity) For $\lambda, m \in \mathbb{R}$ and $\vec{v} \in V$, $\lambda \cdot (m \cdot \vec{v}) = (\lambda m) \cdot \vec{v}$.

7. (distributivity) For $\lambda, m \in \mathbb{R}$ and $\vec{v} \in V$, $(\lambda + m) \cdot \vec{v} = \lambda \cdot \vec{v} + m\vec{v}$.

8. (multiplicative identity) For every $\vec{v} \in V$, $1 \cdot \vec{v} = \vec{v}$.

Example The set of all *n-tuples* of real numbers is a real vector space. We typically denote this $\mathbb{R}^n$. Most of the vectors you've seen in the past have been of this type. For instance, if a position in 2-dimensional space is represented by a vector $\vec{r} \in \mathbb{R}^2$, we can write (in some basis (e^1, e^2)):

$$\vec{r} = r_1 \vec{e}^1 + r_2 \vec{e}^2 \, .$$