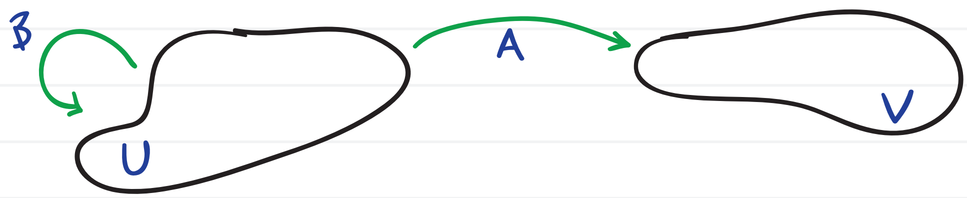
Linear maps + matrices (Adapted from Bullo + Lewis's GCMS)

Linear maps between vector spaces take a vector from one and assign it to a vector in another.



Definition A map $A: U \to V$ between vector spaces $U + V$ is a *linear map* if $A(a\vec{u}) = aA(\vec{u})$ and if $A(\vec{v}+\vec{w}) = A(\vec{v}) + A(\vec{w})$ for each $a \in \mathbb{R}$ and $\vec{u}, \vec{v}, \vec{w} \in U$. If $U = V$, we sometimes call A a *linear transformation*.

Many of the linear maps we consider can be written as *matrices*. The *dimension* of a matrix mapping between two real vector spaces $A: \mathbb{R}^m \to \mathbb{R}^n$ is $n \times m$. We typically write such a matrix in terms of its components in the following manner,

$$A = \underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{n1} & & & A_{nm} \end{bmatrix}}_{m \text{ columns}} \left.\vphantom{\begin{bmatrix} A \\ A \\ A \\ A \end{bmatrix}}\right\} n \text{ rows} .$$

Matrix operating on a vector

Let $\vec{u} \in U$ be a vector. Let $A: U \to V$ be a linear map represented by the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{n1} & & & A_{nm} \end{bmatrix},$$

where $n = \dim(V)$ and $m = \dim(U)$.

Then we write

$$A(\vec{u}) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \ddots & \\ \vdots & & & \\ A_{n1} & & & A_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

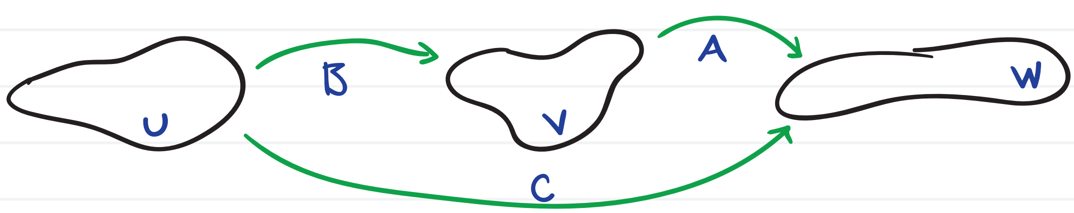
$$= \begin{bmatrix} A_{11}u_1 + A_{12}u_2 + \cdots + A_{1m}u_m \\ A_{21}u_1 + A_{22}u_2 + \cdots + A_{2m}u_m \\ \vdots \\ A_{n1}u_1 + A_{32}u_2 + \cdots + A_{nm}u_m \end{bmatrix} .$$

Example What is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 7 + 2\cdot 8 \\ 3\cdot 7 + 4\cdot 8 \\ 5\cdot 7 + 6\cdot 8 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix} .$$

Matrix composition

Consider vector spaces U, V, W and linear maps $B: U \to V$ and $A: V \to W$.



The composition of maps A and B is the linear map $C: U \to W$. The matrix representations of A and B can be composed by matrix-matrix multiplication in which

$$AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \ddots & \\ \vdots & & & \\ A_{n1} & & & A_{nm} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1l} \\ B_{21} & B_{22} & \ddots & \\ \vdots & & & \\ B_{m1} & & & B_{nl} \end{bmatrix} = \begin{bmatrix} \sum_i A_{1i}B_{i1} & \sum_i A_{1i}B_{i2} & \cdots & \sum_i A_{1i}B_{il} \\ \sum_i A_{2i}B_{i1} & \sum_i A_{2i}B_{i2} & \cdots & \sum_i A_{2i}B_{il} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i A_{ni}B_{i1} & \sum_i A_{ni}B_{i2} & \cdots & \sum_i A_{ni}B_{il} \end{bmatrix}$$

and so $C = AB$. Notice that A has dimension $n \times m$ and B has dimension $m \times \ell$. Therefore C has dimensions $n \times \ell$.

$$n \times m \quad m \times \ell \rightarrow n \times \ell$$

These "inner" dimensions must agree.

Transpose of a linear map

Consider a linear map $A: U \to V$. The transpose of the matrix representation of A is

$$A^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & & \\ \vdots & & \ddots & \\ A_{1m} & & & A_{mn} \end{bmatrix}.$$

Inverse of a linear map

Consider a linear map $A: U \to V$. The inverse of A, denoted $A^{-1}: V \to U$, is defined such that

$$A^{-1}A = AA^{-1} = \mathrm{Id},$$

where Id is the identity map with matrix representation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathrm{Id} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{n \times n}.$$

Note that U and V must have equal dimension (n) in order for an inverse to exist.

Computing the inverse of a linear map is hard, in general. For 2×2 and 3×3 matrices, it isn't too bad. Cramer's Rule is probably the easiest method to use by hand.

Definition Given a square matrix A, Cramer's Rule states that

$$A^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)} .$$

The $\mathrm{Adj}(\cdot)$ and $\det(\cdot)$ functions are the adjugate and determinant. For 2×2 and 3×3 matrices, these are fairly straightforward to compute by hand.

Example Use Cramer's rule to invert $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\mathrm{Adj}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

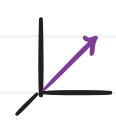
$$\det(A) = ad - bc$$

$$A^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)} = \frac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} .$$

Changing bases

Consider an $\mathbb{R}$-vector space V with bases $(\vec{b}^i) = (\vec{b}^1, \vec{b}^2, \dots, \vec{b}^n)$ + $(\vec{c}^i) = (\vec{c}^1, \vec{c}^2, \dots, \vec{c}^n)$. A vector $\vec{v} \in V$ is a basis-independent object. Therefore,

$$\vec{v} = \sum_{i=1}^{n} [v_b]_i \vec{b}^i = \sum_{i=1}^{n} [v_c]_i \vec{c}^i .$$



If we take the coordinate tuples $\vec{v}_b$ and $\vec{v}_c$, they have a relationship that uniquely defines $B : V \to V$,

$$\vec{v}_c = B \, \vec{v}_b .$$

Sometimes B is called the change of coordinate matrix.

We often consider linear maps $T: V \to V$ that we would like to represent in terms of different bases. Suppose matrix A is a matrix representation of T in basis $(\vec{b}^i \otimes \vec{b}^i)$ and $\hat{A}$ is a matrix representation of T in basis $(\vec{c}^i \otimes \vec{c}^i)$.

If B is the change of coordinate matrix

$$\vec{v}_c = B\,\vec{v}_b,$$

then

$$\hat{A} = B A B^{-1}.$$

This transformation is often called a similarity transformation.

Example Let $\vec{v}_b = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ be a coordinate tuple in the $(\vec{b}^i)$-basis. I.e. we can write $\vec{v} = 1\vec{b}^1 - 3\vec{b}^2$. Let B be a change of coordinate matrix to basis $(\vec{c}^i)$. Then what is the coordinate tuple $\vec{v}_c$. I.e. what are the components of $\vec{v} = [v_c]_1 \vec{c}^1 + [v_c]_2 \vec{c}^2$? Let

$$B = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}.$$

$$\vec{v}_c = B\,\vec{v}_b = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\vec{v} = 1\vec{c}^1 + 5\vec{c}^2 = 1\vec{b}^1 - 3\vec{b}^2.$$

Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ in the $(\vec{b}^i \otimes \vec{b}^i)$-basis. What is its representation in the $(\vec{c}^i \otimes \vec{c}^i)$-basis? $\hat{A}$?

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$$B^{-1} = \frac{\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}}{-2} = \begin{bmatrix} -1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$