1 Mathematics

This chapter describes the foundations of mathematics and why it is so useful to engineers.

1.1 Truth

Before we can discuss mathematical truth, we should begin with a discussion of truth itself.[1] It is important to note that this is obviously extremely incomplete. My aim is to give a sense of the subject via brutal (mis)abbreviation.

Of course, the study of truth cannot but be entangled with the study of the world as such (metaphysics) and of knowledge (epistemology). Some of the following theories presuppose or imply a certain metaphysical and/or epistemological theory, but which these are is controversial.

1.1.1 Neo-Classical Theories of Truth

The neo-classical theories of truth take for granted that there *is* truth and attempt to explain what its precise nature is (Glanzberg 2018). What are provided here are modern understandings of theories developed primarily in the early 20^{th} century.

1.1.1.1 The Correspondence Theory A version of what is called the **correspondence theory** of truth is the following.

> A proposition is true iff there is an existing entity in the world that corresponds with it.

Such existing entities are called **facts**. Facts are relational in that their parts (e.g., subject, predicate, etc.) are related in a certain way.

Under this theory, then, if a proposition does not correspond to a fact, it is **false**.

This theory of truth is rather intuitive and consistently popular (David 2016).

1. For much of this lecture I rely on the thorough overview of (Glanzberg 2018).

1.1.1.2 The Coherence Theory The **coherence theory** of truth is adamant that the truth of any given proposition is only as good as its holistic system of propositions.[2] This includes (but typically goes beyond) a requirement for consistency of a given proposition with the whole and the self-consistency of the whole, itself—sometimes called **coherence**.

For parallelism, let's attempt a succinct formulation of this theory, cast in terms of propositions.

> A proposition is true iff it is has coherence with a system of propositions.

Note that this has no reference to facts, whatsoever. However, it need not necessarily preclude them.

1.1.1.3 The Pragmatic Theory Of the neo-classical theories of truth, this is probably the least agreed upon as having a single clear statement (Glanzberg 2018). However, as with **pragmatism** in general,[3] the pragmatic truth is oriented practically.

Perhaps the most important aspect of this theory is that it is thoroughly a correspondence theory, agreeing that true propositions are those that correspond to the world. However, there is a different focus here that differentiates it from correspondence theory, proper: it values as more true that which has some sort of practical use in human life.

We'll try to summarize pragmatism in two slogans with slightly different emphases; here's the first, again cast in propositional parallel.

> A proposition is true iff it works.[4]

Now, there are two ways this can be understood: (a) the proposition "works" in that it empirically corresponds to the world or (b) the proposition "works" in that it has an effect that some agent intends. The former is pretty standard correspondence theory. The latter is new and fairly obviously has ethical implications, especially today.

Let us turn to a second formulation.

> A proposition is true if it corresponds with a process of inquiry.[5]

This has two interesting facets: (a) an agent's active **inquiry** creates truth and (b) it is a sort of correspondence theory that requires a correspondence of a proposition

2. This is typically put in terms of "beliefs" or "judgments," but for brevity and parallelism I have cast it in terms of propositions. It is to this theory I have probably committed the most violence.

3. Pragmatism was an American philosophical movement of the early 20th century that valued the success of "practical" application of theories. For an introduction, see (Legg and Hookway 2019).

4. This is especially congruent with the work of William James (Legg and Hookway 2019).

5. This is especially congruent with the work of Charles Sanders Peirce (Legg and Hookway 2019).

with a process of inquiry, *not*, as in the correspondence theory, with a fact about the world. The latter has shades of both correspondence theory and coherence theory.

1.1.2 The Picture Theory

Before we delve into this theory, we must take a moment to clarify some terminology.

1.1.2.1 States of Affairs and Facts When discussing the correspondence theory, we have used the term **fact** to mean an actual state of things in the world. A problem arises in the correspondence theory, here. It says that a proposition is true iff there is a fact that corresponds with it. What of a negative proposition like "there are no cows in Antarctica"? We would seem to need a corresponding "negative fact" in the world to make this true. If a fact is taken to be composed of a complex of actual objects and relations, it is hard to imagine such facts.[6]

Furthermore, if a proposition is true, it seems that it is the corresponding fact that makes it so; what, then, makes a proposition false, since there is no fact to support the falsity? (Textor 2016)

And what of nonsense? There are some propositions like "there is a round cube" that are neither true nor false. However, the preceding correspondence theory cannot differentiate between false and nonsensical propositions.

A **state of affairs** is something possible that may or may not be actual (Textor 2016). If a state of affairs is actual, it is said to **obtain**. The picture theory will make central this concept instead of that of the fact.

1.1.2.2 The Picture Theory of Meaning (And Truth) The picture theory of meaning uses the analogy of the **model** or **picture** to explain the meaningfulness of propositions.[7]

> A proposition names possible objects and arranges these names to correspond to a *state of affairs*.

See figure 1.1. This also allows for an easy account of truth, falsity, and nonsense.

6. But (Barker and Jago 2012) have attempted just that.
7. See (Wittgenstein 1922), (Biletzki and Matar 2018), (**glock2016**), and (Dolby 2016).

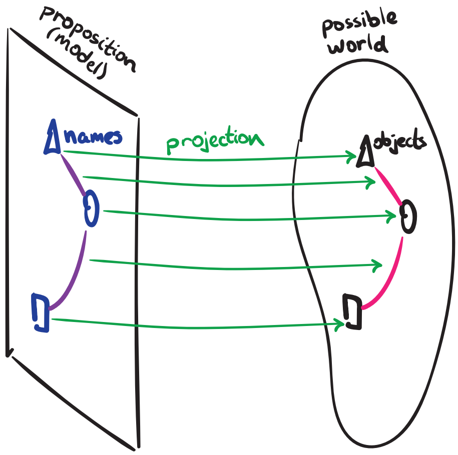

Figure 1.1. A representation of the picture theory.

Nonsense A sentence that appears to be a proposition is actually not if the arrangement of named objects is impossible. Such a sentence is simply **nonsense**.

Truth A proposition is true if the state of affairs it depicts obtains.

Falsity A proposition is false if the state of affairs it depicts does not obtain.

Now, some (Glock 2006) argue this is a correspondence theory and others (Dolby 2016) that it is not. In any case, it certainly solves some issues that have plagued the correspondence theory.

1.1.2.3 "What Cannot Be Said Must Be Shown" Something the picture theory does is declare a limit on what can meaningfully be said. A proposition (as defined above) must be potentially true or false. Therefore, something that cannot be false (something necessarily true) cannot be a proposition (Dolby 2016). And there are certain things that are necessarily true for language itself to be meaningful—paradigmatically, the logical structure of the world. What a proposition does, then, is *show*, via its own logical structure, the *necessary* (for there to be meaningful propositions at all) logical structure of the world.[8]

An interesting feature of this perspective is that it opens up **language itself** to analysis and limitation.[9] And, furthermore, it suggests that the set of what is, is smaller than the set of what can be meaningfully spoken about.

8. See, also, (Žižek 2012), pp. 25-26, from whom I stole the section title.

9. This was one of the contributions to the "linguistic turn" (Wikipedia 2019f) of philosophy in the early 20th century.

1.1.3 The Relativity of Truth

Each subject (i.e., agent) in the world, with their propositions, has a **perspective**: a given moment, a given place, an historical-cultural-linguistic situation. At the very least, the truth of propositions must account for this. Just how a theory of truth should do so is a matter of significant debate (Baghramian and Carter 2019).

Some go so far as to be **skeptical** about truth (Klein 2015), regarding it to be entirely impossible. Others say that while a proposition may or may not be true, we could never come to know this.

Often underlying this conversation is the question of there being a common world in which we all participate, and, if so, whether or not we can properly represent this world in language such that multiple subjects could come to justifiably agree or disagree on the truth of a proposition. If every proposition is so relative that it is relevant to only the proposer, truth would seem of little value. On the other hand, if truth is understood to be "objective"—independent of subjective perspective—a number of objections can be made (Baghramian and Carter 2019), such as that there is no non-subjective perspective from which to judge truth.

1.1.4 Other Ideas about Truth

There are too many credible ideas about truth to attempt a reasonable summary; however, I will attempt to highlight a few important ones.

1.1.4.1 Formal Methods A set of tools was developed for exploring theories of truth, especially correspondence theories.[10] Focus turned from **beliefs** to **sentences**, which are akin to propositions. (Recall that the above theories have already been recast in the more modern language of propositions.) Another aspect of these sentences under consideration is that they begin to be taken as **interpreted sentences**: they are already have meaning.

Beyond this, several technical apparatus are introduced that formalize criteria for truth. For instance, a sentence is given a sign ϕ. A need arises to distinguish between the quotation of sentence ϕ and the unqoted sentence ϕ, which is then given the **quasi-quotation** notation $\ulcorner\phi\urcorner$. For instance, let ϕ stand for *snow is white*; then $\phi \rightarrow$ *snow is white* and $\ulcorner\phi\urcorner \rightarrow$ '*snow is white*'. Tarski introduces **Convention T**, which states that for a fixed language $\boldsymbol{L}$ with fully interpreted sentences, (Glanzberg 2018)

> An adequate theory of truth for $\boldsymbol{L}$ must imply for each sentence ϕ of $\boldsymbol{L}$
> $\ulcorner\phi\urcorner$ is true if and only if ϕ.

Using the same example, then,

10. Especially notable here is the work of Alfred Tarski in the mid-20th century.

> 'snow is white' if and only if snow is white.

Convention T states a general rule for the adequacy of a theory of truth and is used in several contemporary theories.

We can see that these formal methods get quite technical and fun! For more, see (Hodges 2018b; Gómez-Torrente 2019; Hylton and Kemp 2019).

1.1.4.2 Deflationary Theories Deflationary theories of truth try to minimize or eliminate altogether the concept of or use of the term 'truth'. For instance, the **redundancy theory** claim that (Glanzberg 2018):

> To assert that $\ulcorner\phi\urcorner$ is true is just to assert that ϕ.

Therefore, we can eliminate the use of 'is true'.

For more of less, see (Stoljar and Damnjanovic 2014).

1.1.4.3 Language It is important to recognize that language mediates truth; that is, truth is embedded in language. The way language in general affects theories of truth has been studied extensively. For instance, whether the **truth-bearer** is a belief or a proposition or a sentence—or something else—has been much discussed. The importance of the **meaning** of truth-bearers like sentences has played another large role. Theories of meaning, like the picture theory presented above, are often closely intertwined with theories of truth.

One of the most popular theories of meaning is called the **theory of use**:

> For a large class of cases of the employment of the word "meaning" – though not for all – this word can be explained in this way: the meaning of a word is its use in the language. (Wittgenstein, Hacker, and Schulte 2010)

This theory is accompanied by the concept of **language-games**, which are loosely defined as rule-based contexts within which sentences have uses. The idea is that the meaning of a given sentence is its use in a network of meaning that is constantly evolving. This view tends to be understood as deflationary or relativistic about truth.

1.1.4.4 Metaphysical and Epistemological Considerations We began with the recognition that truth is intertwined with metaphysics and epistemology. Let's consider a few such topics.

The first is **metaphysical realism**, which claims that there is a world existing objectively: independently of how we think about or describe it. This "realism" tends to be closely tied to, yet distinct from, **scientific realism**, which goes further, claiming the world is "actually" as science describes, independently of the scientific descriptions (e.g., there are actual objects corresponding to the phenomena we call atoms, molecules, light particles, etc.).

There have been many challenges to the realist claim (for some recent versions, see (Khlentzos 2016)) put forth by what is broadly called **anti-realism**. These vary, but often challenge the ability of realists to properly link language to supposed objects in the world.

Metaphysical idealism has been characterized as claiming that "mind" or "subjectivity" generate or completely compose the world, which has no being outside mind. **Epistemological idealism**, on the other hand, while perhaps conceding that there is a world independent of mind, claims all knowledge of the world is created through mind and for mind and therefore can never escape a sort of mind-world gap.[11] This epistemological idealism has been highly influential since the work of Immanuel Kant (Kant, Guyer, and Wood 1999) in the late 18th century, which ushered in the idea of the **noumenal world** in-itself and the **phenomenal world**, which is how the noumenal world presents to us. Many have held that phenomena can be known through inquiry, whereas noumena are inaccessible. Furthermore, what can be known is restricted by the categories pre-existent in the knower.

Another approach, taken by Georg Wilhelm Friedrich Hegel (Redding 2018) and other German idealists following Kant, is to reframe reality as thoroughly integrating subjectivity (Hegel and Miller 1998; Žižek 2012); that is, "everything turns on grasping and expressing the True, not only as *Substance*, but equally as *Subject*." A subject's proposition is true inasmuch as it corresponds with its **Notion** (approximately: the idea or meaning for the subject). Some hold that this idealism is compatible with a sort of metaphysical realism, at least as far as understanding is not independent of but rather beholden to reality (Žižek 2012; p. 906 ff.).

Clearly, all these ideas have many implications for theories of truth and vice versa.

1.1.5 Where This Leaves Us

The truth is hard. What may at first appear to be a simple concept becomes complex upon analysis. It is important to recognize that we have only sampled some highlights of the theories of truth. I recommend further study of this fascinating topic.

Despite the difficulties of finding definitive grounds for understanding truth, we are faced with the task of provisionally forging ahead. Much of what follows in the study of mathematics makes certain implicit and explicit assumptions about truth. However, we have found that the foundations of these assumptions may themselves be problematic. It is my contention that, despite the lack of clear foundations, *it is still worth studying engineering analysis, its mathematical foundations, and the foundations*

11. These definitions are explicated by (Guyer and Horstmann 2018).

of truth itself. My justification for this claim is that I find the utility and the beauty of this study highly rewarding.

1.2 The Foundations of Mathematics

Mathematics has long been considered exemplary for establishing truth. Primarily, it uses a method that begins with **axioms**—unproven propositions that include undefined terms—and uses logical **deduction** to **prove** other propositions (**theorems**): to show that they are necessarily true if the axioms are.

It may seem obvious that truth established in this way would always be relative to the truth of the axioms, but throughout history this footnote was often obscured by the "obvious" or "intuitive" universal truth of the axioms.[12] For instance, **Euclid** (Wikipedia 2019c) founded **geometry**—the study of mathematical objects traditionally considered to represent physical space, like points, lines, etc.—on axioms thought so solid that it was not until the early 19th century that **Carl Friedrich Gauss** (Wikipedia 2019b) and others recognized this was only one among many possible geometries (Kline 1982) resting on different axioms. Furthermore, **Aristotle** (Shields 2016) had acknowledged that reasoning must begin with undefined terms; however, even Euclid (presumably aware of Aristotle's work) seemed to forget this and provided definitions, obscuring the foundations of his work and starting mathematics on a path that for over 2,000 years would forget its own relativity (Kline 1982; p. 101-2).

The foundations of Euclid were even shakier than its murky starting point: several unstated axioms were used in proofs and some proofs were otherwise erroneous. However, for two millennia, mathematics was seen as the field wherein truth could be established beyond doubt.

1.2.1 Algebra *Ex nihilo*

Although not much work new geometry appeared during this period, the field of **algebra** (Wikipedia 2019a)—the study of manipulations of symbols standing for numbers in general—began with no axiomatic foundation whatsoever. The Greeks had a notion of **rational numbers**, ratios of **natural numbers** (positive **integers**), and it was known that many solutions to algebraic equations were **irrational** (could not be expressed as a ratio of integers). But these irrational numbers, like virtually everything else in algebra, were gradually accepted because they were so useful in solving practical problems (they could be approximated by rational numbers and this seemed good enough). The rules of basic arithmetic were accepted as applying

12. Throughout this section, for the history of mathematics I rely heavily on (Kline 1982).