*of truth itself.* My justification for this claim is that I find the utility and the beauty of this study highly rewarding.

## 1.2 The Foundations of Mathematics

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Mathematics has long been considered exemplary for establishing truth. Primarily, it uses a method that begins with **axioms**—unproven propositions that include undefined terms—and uses logical **deduction** to **prove** other propositions (**theorems**): to show that they are necessarily true if the axioms are.

It may seem obvious that truth established in this way would always be relative to the truth of the axioms, but throughout history this footnote was often obscured by the "obvious" or "intuitive" universal truth of the axioms.[12] For instance, **Euclid** (Wikipedia 2019c) founded **geometry**—the study of mathematical objects traditionally considered to represent physical space, like points, lines, etc.—on axioms thought so solid that it was not until the early 19th century that **Carl Friedrich Gauss** (Wikipedia 2019b) and others recognized this was only one among many possible geometries (Kline 1982) resting on different axioms. Furthermore, **Aristotle** (Shields 2016) had acknowledged that reasoning must begin with undefined terms; however, even Euclid (presumably aware of Aristotle's work) seemed to forget this and provided definitions, obscuring the foundations of his work and starting mathematics on a path that for over 2,000 years would forget its own relativity (Kline 1982; p. 101-2).

The foundations of Euclid were even shakier than its murky starting point: several unstated axioms were used in proofs and some proofs were otherwise erroneous. However, for two millennia, mathematics was seen as the field wherein truth could be established beyond doubt.

### 1.2.1 Algebra *Ex nihilo*

Although not much work new geometry appeared during this period, the field of **algebra** (Wikipedia 2019a)—the study of manipulations of symbols standing for numbers in general—began with no axiomatic foundation whatsoever. The Greeks had a notion of **rational numbers**, ratios of **natural numbers** (positive **integers**), and it was known that many solutions to algebraic equations were **irrational** (could not be expressed as a ratio of integers). But these irrational numbers, like virtually everything else in algebra, were gradually accepted because they were so useful in solving practical problems (they could be approximated by rational numbers and this seemed good enough). The rules of basic arithmetic were accepted as applying

12. Throughout this section, for the history of mathematics I rely heavily on (Kline 1982).

to these and other forms of new numbers that arose in algebraic solutions: **negative**, **imaginary**, and **complex numbers**.

### 1.2.2 The Application of Mathematics to Science

During this time, mathematics was being applied to **optics** and **astronomy**. Sir Isaac Newton then built **calculus** upon algebra, applying it to what is now known as **Newtonian mechanics**, which was really more the product of Leonhard Euler (Smith 2008; Wikipedia 2019e). Calculus introduced its own dubious operations, but the success of mechanics in describing and predicting physical phenomena was astounding. Mathematics was hailed as the language of God (later, Nature).

### 1.2.3 The Rigorization of Mathematics

It was not until Gauss created **non-Euclidean geometry**, in which Euclid's were shown to be one of many possible axioms compatible with the world, and William Rowan Hamilton (Wikipedia 2019k) created **quaternions** (Wikipedia 2019i), a number system in which multiplication is noncommunicative, that it became apparent something was fundamentally wrong with the way truth in mathematics had been understood. This started a period of **rigorization** in mathematics that set about axiomatizing and proving 19$^{th}$ century mathematics. This included the development of **symbolic logic**, which aided in the process of deductive reasoning.

An aspect of this rigorization is that mathematicians came to terms with the axioms that include undefined terms. For instance, a "point" might be such an undefined term in an axiom. A **mathematical model** is what we create when we attach these undefined terms to objects, which can be anything consistent with the axioms.[13] The system that results from proving theorems would then apply to anything "properly" described by the axioms. So two masses might be assigned "points" in a Euclidean geometric space, from which we could be confident that, for instance, the "distance" between these masses is the Euclidean norm of the line drawn between the points. It could be said, then, that a "point" in Euclidean geometry is **implicitly defined** by its axioms and theorems, and nothing else. That is, mathematical objects are *not* inherently tied to the physical objects to which we tend to apply them. Euclidean geometry is not the study of physical space, as it was long considered—it is the study of the objects implicitly defined by its axioms and theorems.

13. The branch of mathematics called *model theory* concerns itself with general types of models that can be made from a given formal system, like an axiomatic mathematical system. For more on model theory, see (Hodges 2018a). It is noteworthy that the engineering/science use of the term "mathematical model" is only loosely a "model" in the sense of model theory.

### 1.2.4 The Foundations of Mathematics Are Built

The building of the modern foundations mathematics began with clear axioms, solid reasoning (with symbolic logic), and lofty yet seemingly attainable goals: prove theorems to support the already ubiquitous mathematical techniques in geometry, algebra, and calculus from axioms; furthermore, prove that these axioms (and things they imply) do not contradict each other, i.e., are **consistent**, and that the axioms are not results of each other (one that can be derived from others is a **theorem**, not an axiom).

**Set theory** is a type of formal axiomatic system that all modern mathematics is expressed with, so set theory is often called the **foundation** of mathematics (Bagaria 2019). We will study the basics in (**ch:set_theory**). The primary objects in set theory are **sets**: informally, collections of mathematical objects. There is not just one a single set of axioms that is used as the foundation of all mathematics for reasons will review in a moment. However, the most popular set theory is **Zermelo-Fraenkel set theory with the axiom of choice** (ZFC). The axioms of ZF sans C are as follows. (Bagaria 2019)

**Extensionality** If two sets $A$ and $B$ have the same elements, then they are equal.

**Empty set** There exists a set, denoted by $\varnothing$ and called the empty set, which has no elements.

**Pair** Given any sets $A$ and $B$, there exists a set, denoted by $\{A, B\}$, which contains $A$ and $B$ as its only elements. In particular, there exists the set $\{A\}$ which has $A$ as its only element.

**Power set** For every set $A$ there exists a set, denoted by $\mathcal{P}(A)$ and called the power set of $A$, whose elements are all the subsets of $A$.

**Union** For every set $A$, there exists a set, denoted by $\bigcup A$ and called the union of $A$, whose elements are all the elements of the elements of $A$.

**Infinity** There exists an infinite set. In particular, there exists a set $Z$ that contains $\varnothing$ and such that if $A \in Z$, then $\bigcup\{A, \{A\}\} \in Z$.

**Separation** For every set $A$ and every given property, there is a set containing exactly the elements of $A$ that have that property. A property is given by a formula $\varphi$ of the first-order language of set theory. Thus, separation is not a single axiom but an axiom schema, that is, an infinite list of axioms, one for each formula $\varphi$.

**Replacement** For every given definable function with domain a set $A$, there is a set whose elements are all the values of the function.

ZFC also has the axiom of choice. (Bagaria 2019)

**Choice** For every set $A$ of pairwise-disjoint non-empty sets, there exists a set that contains exactly one element from each set in $A$.

### 1.2.5 The Foundations Have Cracks

The foundationalists' goal was to prove that some set of axioms from which all of mathematics can be derived is both consistent (contains no contradictions) and complete (every true statement is provable). The work of Kurt Gödel (Kennedy 2018) in the mid 20$^{th}$ century shattered this dream by proving in his **first incompleteness theorem** that any consistent formal system within which one can do some amount of basic arithmetic is **incomplete**! His argument is worth reviewing (see (Raatikainen 2018)), but at its heart is an **undecidable** statement like "This sentence is unprovable." Let $U$ stand for this statement. If it is true it is unprovable. If it is provable it is false. Therefore, it is true iff it is provable. Then he shows that if a statement $A$ that essentially says "arithmetic is consistent" is provable, then so is the undecidable statement $U$. But if $U$ is to be consistent, it cannot be provable, and, therefore neither can $A$ be provable!

This is problematic. It tells us virtually any conceivable axiomatic foundation of mathematics is incomplete. If one is complete, it is inconsistent (and therefore worthless). One problem this introduces is that a true theorem may be impossible to prove; but, it turns out, we can never know that in advance of its proof if it is provable.

But it gets worse: Gödel's **second incompleteness theorem** shows that such systems cannot even be shown to be consistent! This means, at any moment, someone could find an inconsistency in mathematics, and not only would we lose some of the theorems: we would lose them all. This is because, by what is called the **material implication** (Kline 1982; pp. 187-8 264), if one contradiction can be found, *every proposition can be proven* from it. And if this is the case, all (even proven) theorems in the system would be suspect.

Even though no contradiction has yet appeared in ZFC, its axiom of choice, which is required for the proof of most of what has thus far been proven, generates the **Banach-Tarski paradox** that says a sphere of diameter $x$ can be partitioned into a finite number of pieces and recombined to form *two* spheres of diameter $x$. Troubling, to say the least! Attempts were made for a while to eliminate the use of the axiom of choice, but our buddy Gödel later proved that if ZF is consistent, so is ZFC ( p. 267).

### 1.2.6 Mathematics Is Considered Empirical

Since its inception, mathematics has been applied extensively to the modeling of the world. Despite its cracked foundations, it has striking utility. Many recent leading minds of mathematics, philosophy, and science suggest we treat mathematics as **empirical**, like any science, subject to its success in describing and predicting events in the world. As (Kline 1982) summarizes,

> The upshot [...] is that sound mathematics must be determined not by any one foundation which may some day prove to be right. The "correctness" of mathematics must be judged by its application to the physical world. Mathematics is an empirical science much as Newtonian mechanics. It is correct only to the extent that it works and when it does not, it must be modified. It is not a priori knowledge even though it was so regarded for two thousand years. It is not absolute or unchangeable.

## 1.3 Problems

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