2 Mathematical Reasoning, Logic, and Set Theory

In order to communicate mathematical ideas effectively, **formal languages** have been developed within which **logic**, i.e. deductive (mathematical) **reasoning**, can proceed. **Propositions** are statements that can be either true $\top$ or false $\bot$. Axiomatic systems begin with statements (axioms) assumed true. **Theorems** are **proven** by deduction. In many forms of logic, like **propositional calculus** (Wikipedia 2019h), compound propositions are constructed via **logical connectives** like "and" and "or" of atomic propositions (see section 2.2). In others, like **first-order logic** (Wikipedia 2019d), there are also logical **quantifiers** like "for every" and "there exists."

The mathematical objects and operations about which most propositions are made are expressed in terms of **set theory**, which was introduced in section 1.2 and will be expanded upon in section 2.1. We can say that mathematical reasoning is comprised of mathematical objects and operations expressed in set theory and logic allows us to reason therewith.

2.1 Introduction to Set Theory

Set theory is the language of the modern foundation of mathematics, as discussed in chapter 1. It is unsurprising, then, that it arises throughout the study of mathematics. We will use set theory extensively in chapter 3 on probability theory.

The axioms of ZFC set theory were introduced in chapter 1. Instead of proceeding in the pure mathematics way of introducing and proving theorems, we will opt for a more applied approach in which we begin with some simple definitions and include basic operations. A more thorough and still readable treatment is given by (Ciesielski 1997) and a very gentle version by (Enderton 1977).

A **set** is a collection of objects. Set theory gives us a way to describe these collections. Often, the objects in a set are numbers or sets of numbers. However, a set could represent collections of zebras and trees and hairballs. For instance, here are some sets:

$$\{1, 5, \pi\} \quad \{\text{zebra named "Calvin"}, \text{ a burnt cheeto}\} \quad \{\{1, 2\}, \{5, \text{hippo}, 7\}, 62\}.$$

A **field** is a set with special structure. This structure is provided by the **addition** (+) and **multiplication** (×) operators and their inverses **subtraction** (−) and **division** (÷). The quintessential example of a field is the set of **real numbers** $\mathbb{R}$, which admits these operators, making it a field. The reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, the integers $\mathbb{Z}$, and the natural numbers[1] $\mathbb{N}$ are the fields we typically consider.

Set membership is the belonging of an object to a set. It is denoted with the symbol $\in$, which can be read "is an element of," for element x and set X:

$$x \in X.$$

For instance, we might say $7 \in \{1, 7, 2\}$ or $4 \notin \{1, 7, 2\}$. Or, we might declare that a is a real number by stating: $x \in \mathbb{R}$.

Set operations can be used to construct new sets from established sets. We consider a few common set operations, now.

The **union** $\cup$ of sets is the set containing all the elements of the original sets (no repetition allowed). The union of sets A and B is denoted $A \cup B$. For instance, let $A = \{1, 2, 3\}$ and $B = \{-1, 3\}$; then

$$A \cup B = \{1, 2, 3, -1\}.$$

The **intersection** $\cap$ of sets is a set containing the elements common to all the original sets. The intersection of sets A and B is denoted $A \cap B$. For instance, let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$; then

$$A \cap B = \{2, 3\}.$$

If two sets have no elements in common, the intersection is the **empty set** $\varnothing = \{\}$, the unique set with no elements.

The **set difference** of two sets A and B is the set of elements in A that aren't also in B. It is denoted $A \setminus B$. For instance, let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Then

$$A \setminus B = \{1\} \quad B \setminus A = \{4\}.$$

A **subset** $\subseteq$ of a set is a set, the elements of which are contained in the original set. If the two sets are equal, one is still considered a subset of the other. We call a subset that is not equal to the other set a **proper subset** $\subset$. For instance, let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Then

$$B \subseteq A \quad B \subset A \quad A \subseteq A.$$

1. When the natural numbers include zero, we write $\mathbb{N}_0$.

The **complement** of a subset is a set of elements of the original set that aren't in the subset. For instance, if $B \subseteq A$, then the complement of B, denoted $\overline{B}$ is

$$\overline{B} = A \setminus B.$$

The **cartesian product** of two sets A and B is denoted $A \times B$ and is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. It's worthwhile considering the following notation for this definition:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

which means "the cartesian product of A and B is the ordered pair (a, b) such that $a \in A$ and $b \in B$" in **set-builder notation** (Wikipedia 2019j).

Let A and B be sets. A **map** or **function** f from A to B is an assignment of some element $a \in A$ to each element $b \in B$. The function is denoted $f : A \to B$ and we say that f maps each element $a \in A$ to an element $f(a) \in B$ called the **value** of a under f, or $a \mapsto f(a)$. We say that f has **domain** A and **codomain** B. The **image** of f is the subset of its codomain B that contains the values of all elements mapped by f from its domain A.

2.2 Logical Connectives and Quantifiers

In order to make compound propositions, we need to define logical connectives. In order to specify quantities of variables, we need to define logical quantifiers. The following is a form of **first-order logic** (Wikipedia 2019d).

2.2.1 Logical Connectives

A proposition can be either true $\top$ and false $\bot$. When it does not contain a logical connective, it is called an **atomistic proposition**. To combine propositions into a **compound proposition**, we require **logical connectives**. They are **not** ($\neg$), **and** ($\wedge$), and **or** ($\vee$). Table 2.1 is a **truth table** for a number of connectives.

Table 2.1: a truth table for logical connectives. The first two columns are the truth values of propositions p and q; the rest are *outputs*.

		not	and	or	nand	nor	xor	xnor
p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \uparrow q$	$p \downarrow q$	$p \veebar q$	$p \Leftrightarrow q$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\top$