The **complement** of a subset is a set of elements of the original set that aren't in the subset. For instance, if $B \subseteq A$, then the complement of $B$, denoted $\overline{B}$ is

$$\overline{B} = A \setminus B.$$

The **cartesian product** of two sets $A$ and $B$ is denoted $A \times B$ and is the set of all ordered pairs $(a, b)$ where $a \in A$ and $b \in B$. It's worthwhile considering the following notation for this definition:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

which means "the cartesian product of $A$ and $B$ is the ordered pair $(a, b)$ such that $a \in A$ and $b \in B$" in **set-builder notation** (Wikipedia 2019j).

Let $A$ and $B$ be sets. A **map** or **function** $f$ from $A$ to $B$ is an assignment of some element $a \in A$ to each element $b \in B$. The function is denoted $f : A \to B$ and we say that $f$ maps each element $a \in A$ to an element $f(a) \in B$ called the **value** of $a$ under $f$, or $a \mapsto f(a)$. We say that $f$ has **domain** $A$ and **codomain** $B$. The **image** of $f$ is the subset of its codomain $B$ that contains the values of all elements mapped by $f$ from its domain $A$.

## 2.2 Logical Connectives and Quantifiers

In order to make compound propositions, we need to define logical connectives. In order to specify quantities of variables, we need to define logical quantifiers. The following is a form of **first-order logic** (Wikipedia 2019d).

### 2.2.1 Logical Connectives

A proposition can be either true $\top$ and false $\bot$. When it does not contain a logical connective, it is called an **atomistic proposition**. To combine propositions into a **compound proposition**, we require **logical connectives**. They are **not** ($\neg$), **and** ($\wedge$), and **or** ($\vee$). Table 2.1 is a **truth table** for a number of connectives.

Table 2.1: a truth table for logical connectives. The first two columns are the truth values of propositions $p$ and $q$; the rest are *outputs*.

| | | not | and | or | nand | nor | xor | xnor |
|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\neg p$ | $p \wedge q$ | $p \vee q$ | $p \uparrow q$ | $p \downarrow q$ | $p \veebar q$ | $p \Leftrightarrow q$ |
| $\bot$ | $\bot$ | $\top$ | $\bot$ | $\bot$ | $\top$ | $\top$ | $\bot$ | $\top$ |
| $\bot$ | $\top$ | $\top$ | $\bot$ | $\top$ | $\top$ | $\bot$ | $\top$ | $\bot$ |
| $\top$ | $\bot$ | $\bot$ | $\bot$ | $\top$ | $\top$ | $\bot$ | $\top$ | $\bot$ |
| $\top$ | $\top$ | $\bot$ | $\top$ | $\top$ | $\bot$ | $\bot$ | $\bot$ | $\top$ |

### 2.2.2 Quantifiers

Logical quantifiers allow us to indicate the quantity of a variable. The **universal quantifier symbol** $\forall$ means "for all". For instance, let $A$ be a set; then $\forall a \in A$ means "for all elements in $A$" and gives this quantity variable $a$. The **existential quantifier** $\exists$ means "there exists at least one" or "for some". For instance, let $A$ be a set; then $\exists a \in A \ldots$ means "there exists at least one element $a$ in $A$ …."

## 2.3 Problems

H4

**Problem 2.1** HARDHAT For the following, write the set described in set-builder notation.

a. $A = \{2, 3, 5, 9, 17, 33, \cdots\}$.
b. $B$ is the set of integers divisible by 11.
c. $C = \{1/3, 1/4, 1/5, \cdots\}$.
d. $D$ is the set of reals between $-3$ and 42.

**Problem 2.2** ANATOMY Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Prove the *Cauchy-Schwarz Inequality*

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\|. \tag{2.1}$$

Hint: you may find the geometric definition of the dot product helpful.

**Problem 2.3** ACOUSTIC Let $\boldsymbol{x} \in \mathbb{R}^n$. Prove that

$$\boldsymbol{x} \cdot \boldsymbol{x} = \|\boldsymbol{x}\|^2. \tag{2.2}$$

Hint: you may find the geometric definition of the dot product helpful.

**Problem 2.4** SUSANNA Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Prove the *Triangle Inequality*

$$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|. \tag{2.3}$$

Hint: you may find the Cauchy-Schwarz Inequality helpful.