3 Probability and Random Processes

This chapter introduces probability and random variables. Important in itself, it will also provide the basis for statistics, described in chapter 4.

3.1 Probability and Measurement

Probability theory is a well-defined branch of mathematics. Andrey Kolmogorov described a set of axioms in 1933 that are still in use today as the foundation of probability theory.[1]

We will implicitly use these axioms in our analysis. The **interpretation** of probability is a contentious matter. Some believe probability quantifies the frequency of the occurrence of some **event** that is repeated in a large number of trials. Others believe it quantifies the state of our knowledge or belief that some event will occur.

In experiments, our measurements are tightly coupled to probability. This is apparent in the questions we ask. Here are some examples.

1. How common is a given event?
2. What is the probability we will reject a good theory based on experimental results?
3. How repeatable are the results?
4. How confident are we in the results?
5. What is the character of the fluctuations and drift in the data?
6. How much data do we need?

1. For a good introduction to probability theory, see (Ash 2008) or (Jaynes et al. 2003).

3.2 Basic Probability Theory

The mathematical model for a class of measurements is called the **probability space** and is composed of a mathematical triple of a sample space Ω, σ-algebra $\mathcal{F}$, and probability measure P, typically denoted $(\Omega, \mathcal{F}, P)$, each of which we will consider in turn (Wikipedia 2019g).

The **sample space** Ω of an experiment is the set representing all possible **outcomes** of the experiment. If a coin is flipped, the sample space is $\Omega = \{H, T\}$, where H is *heads* and T is *tails*. If a coin is flipped twice, the sample space could be

$$\Omega = \{HH, HT, TH, TT\}.$$

However, *the same experiment can have different sample spaces*. For instance, for two coin flips, we could also choose

$$\Omega = \{\text{the flips are the same, the flips are different}\}.$$

We base our choice of Ω on the problem at hand.

An **event** is a subset of the sample space. That is, an event corresponds to a yes-or-no question about the experiment. For instance, event A (remember: $A \subseteq \Omega$) in the coin flipping experiment (two flips) might be $A = \{HT, TH\}$. A is an event that corresponds to the question, "Is the second flip different than the first?" A is the event for which the answer is "yes."

3.2.1 Algebra of Events

Because events are sets, we can perform the usual set operations with them.

Example 3.1

Consider a toss of a single die. We choose the sample space to be $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let the following define events.

$$A \equiv \{\text{the result is even}\} = \{2, 4, 6\}$$

$$B \equiv \{\text{the result is greater than 2}\} = \{3, 4, 5, 6\}.$$

Find the following event combinations:

$$A \cup B \quad A \cap B \quad A \setminus B \quad B \setminus A \quad \overline{A} \setminus B.$$

$$A \cup B = \{2, 3, 4, 5, 6\} \quad \text{(even or greater than 2)}$$

$$A \cap B = \{4, 6\} \quad \text{(even and greater than 2)}$$

$$A \setminus B = \{2\} \quad \text{(even but not greater than 2)}$$

$$B \setminus A = \{3, 5\} \quad \text{(greater than two and odd)}$$

$$\overline{A} \setminus B = \{1, 3, 5\} \setminus \{3, 4, 5, 6\} \quad \text{(not even and not greater than 2).}$$