3.5 Random Variables

Probabilities are useful even when they do not deal strictly with events. It often occurs that we measure something that has randomness associated with it. We use random variables to represent these measurements.

A **random variable** $X : \Omega \to \mathbb{R}$ is a function that maps an outcome ω from the sample space Ω to a real number $x \in \mathbb{R}$, as shown in figure 3.2. A random variable will be denoted with a capital letter (e.g. X and K) and a specific value that it maps to (the value) will be denoted with a lowercase letter (e.g. x and k).

A **discrete random variable** K is one that takes on discrete values. A **continuous random variable** X is one that takes on continuous values.

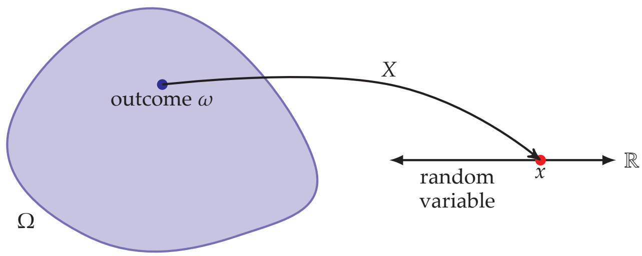

Figure 3.2. A random variable X maps an outcome $\omega \in \Omega$ to an $x \in \mathbb{R}$.

Example 3.5

Roll two unbiased dice. Let K be a random variable representing the sum of the two. Let $P(k)$ be the probability of the result $k \in K$. Plot and interpret $P(k)$.

Figure 3.3 shows the probability of each sum occurring.

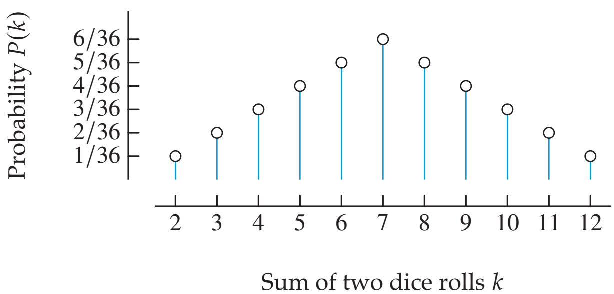

Figure 3.3. PMF for the summ of two dice rolled.

We call this a **probability mass function**. It tells us the probability with wich each outcome will occur.

Example 3.6

A resistor at nonzero temperature without any applied voltage exhibits an interesting phenomenon: its voltage randomly fluctuates. This is called *Johnson-Nyquist noise* and is a result of *thermal excitation* of charge carriers (electrons, typically). For a given resistor and measurement system, let the *probability density function* f_V of the voltage V across an unrealistically hot resistor be

$$f_V(V) = \frac{1}{\sqrt{\pi}} e^{-V^2}.$$

Plot and interpret the meaning of this function.

The PDF is shown in figure 3.4.

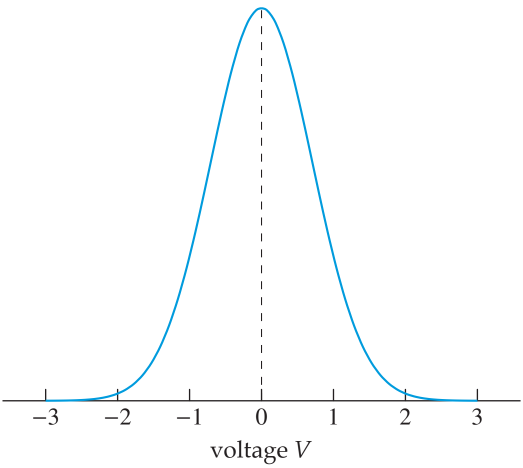

Figure 3.4. The probability density function.

A probability density function must be integrated to find probability. The probability a randomly measured voltage will be between two voltages is the integral of f_V across that voltage interval. Note that a resistor would need to be extremely hot to have such a large thermal noise. In the next lecture, we consider more probability density functions.

3.6 Probability Density and Mass Functions

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Consider an experiment that measures a random variable. We can plot the relative frequency of the measurand landing in different "bins" (ranges of values). This is called a **frequency distribution** or a **probability mass function** (PMF).

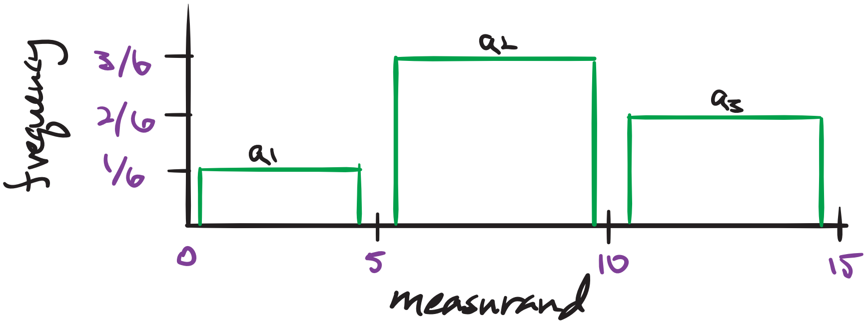

Figure 3.5. Plot of a probability mass function.

Consider, for instance, a probability mass function as plotted in figure 3.5, where a frequency a_i can be interpreted as an estimate of the probability of the measurand being in the ith interval. The sum of the frequencies must be unity:

$$\sum_{i=1}^{k} a_i = 1$$

with k being the number of bins.

The **frequency density distribution** is similar to the frequency distribution, but with $a_i \mapsto a_i/\Delta x$, where Δx is the bin width.

If we let the bin width approach zero, we derive the **probability density function** (PDF)

$$f(x) = \lim_{\substack{k\to\infty \\ \Delta x\to 0}} \sum_{j=1}^{k} a_j/\Delta x.$$

We typically think of a probability density function f, like the one in figure 3.6 as a function that can be integrated over to find the probability of the random variable (measurand) being in an interval $[a, b]$:

$$P(x \in [a, b]) = \int_a^b f(\chi)d\chi.$$