3.6 Probability Density and Mass Functions

9Z

Consider an experiment that measures a random variable. We can plot the relative frequency of the measurand landing in different "bins" (ranges of values). This is called a **frequency distribution** or a **probability mass function** (PMF).

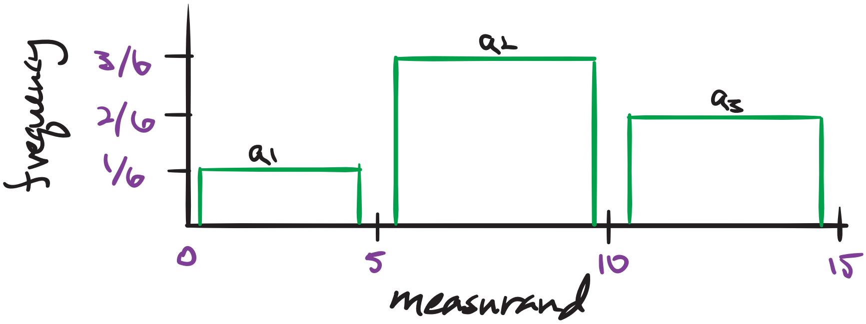

Figure 3.5. Plot of a probability mass function.

Consider, for instance, a probability mass function as plotted in figure 3.5, where a frequency a_i can be interpreted as an estimate of the probability of the measurand being in the ith interval. The sum of the frequencies must be unity:

$$\sum_{i=1}^{k} a_i = 1$$

with k being the number of bins.

The **frequency density distribution** is similar to the frequency distribution, but with $a_i \mapsto a_i/\Delta x$, where Δx is the bin width.

If we let the bin width approach zero, we derive the **probability density function** (PDF)

$$f(x) = \lim_{\substack{k \to \infty \\ \Delta x \to 0}} \sum_{j=1}^{k} a_j/\Delta x.$$

We typically think of a probability density function f, like the one in figure 3.6 as a function that can be integrated over to find the probability of the random variable (measurand) being in an interval $[a, b]$:

$$P(x \in [a, b]) = \int_a^b f(\chi) d\chi.$$

Of course,

$$P(x \in (-\infty, \infty)) = \int_{-\infty}^{\infty} f(\chi) d\chi$$
$$= 1.$$

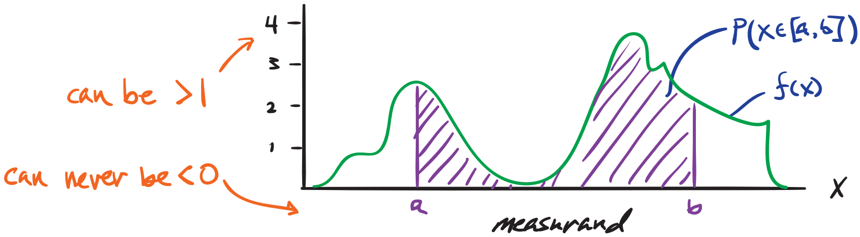

Figure 3.6. Plot of a probability density function.

We now consider a common PMF and a common PDF.

3.6.1 Binomial PMF

Consider a random binary sequence of length n such that each element is a random 0 or 1, generated independently, like

$$(1, 0, 1, 1, 0, \cdots, 1, 1).$$

Let events $\{1\}$ and $\{0\}$ be mutually exclusive and exhaustive and $P(\{1\}) = p$. The probability of the sequence above occurring is

$$P((1, 0, 1, 1, 0, \cdots, 1, 1)) = p(1-p)pp(1-p)\cdots pp.$$

There are n choose k,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

possible combinations of k ones for n bits. Therefore, the probability of any combination of k ones in a series is

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

We call section 3.6.1 the **binomial distribution PDF**.

Example 3.7

Consider a field sensor that fails for a given measurement with probability p. Given n measurements, plot the binomial PMF as a function of k failed measurements for a few different probabilities of failure $p \in [0.04, 0.25, 0.5, 0.75, 0.96]$.

listing 3.1 shows Python code for constructing the PDFs plotted in figure 3.7. Note that the symmetry is due to the fact that events $\{1\}$ and $\{0\}$ are mutually exclusive and exhaustive.

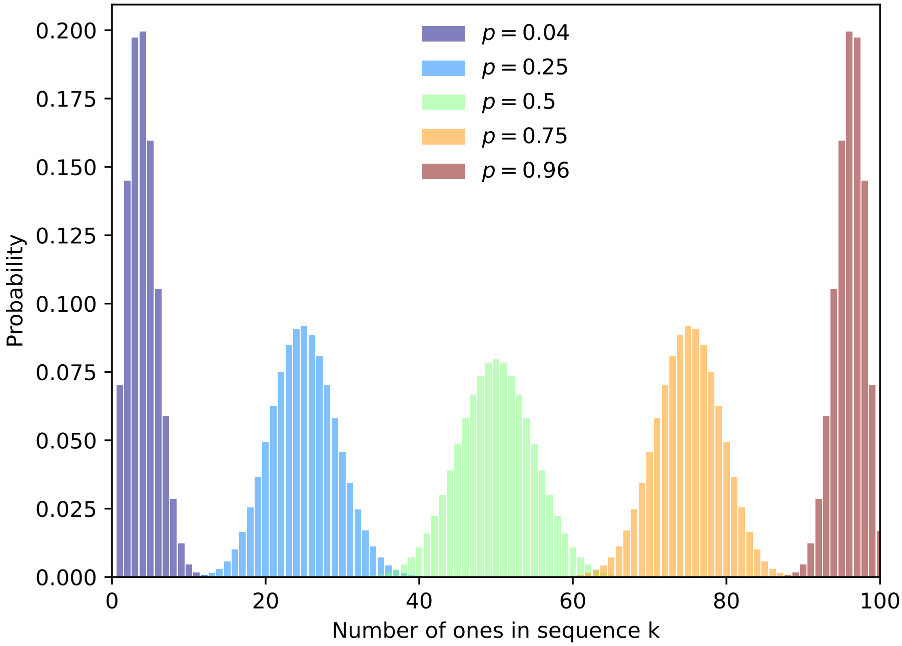

Figure 3.7. Binomial PDF for $n = 100$ measurements and different values of $P(\{1\}) = p$, the probability of a measurement error. The plot is generated by the Python code of **??**.

Listing 3.1 Python code that generates the binomial PDF

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import comb

# Parameters
n = 100
k_a = np.arange(1, n + 1)
p_a = np.array([0.04, 0.25, 0.5, 0.75, 0.96])

# Binomial function
def binomial(n, k, p):
    return comb(n, k) * (p ** k) * ((1 - p) ** (n - k))

# Constructing the array
f_a = np.zeros((len(k_a), len(p_a)))
for i in range(len(k_a)):
    for j in range(len(p_a)):
        f_a[i, j] = binomial(n, k_a[i], p_a[j])

# Plot
plt.figure()
colors = plt.cm.jet(np.linspace(0, 1, len(p_a)))
for j in range(len(p_a)):
    plt.bar(k_a, f_a[:, j], color=colors[j], alpha=0.5, label=f'$p =
    ↪ {p_a[j]}$')

plt.legend(loc='best', frameon=False, fontsize='medium')
plt.xlabel('Number of ones in sequence k')
plt.ylabel('Probability')
plt.xlim([0, 100])
plt.show()

# Save the plot to pdf
plt.savefig('binomial-pdf.pdf', bbox_inches='tight')
```

3.6.2 Gaussian PDF

The **Gaussian** or *normal random variable* x has PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{-(x-\mu)^2}{2\sigma^2}.$$

Although we're not quite ready to understand these quantities in detail, it can be shown that the parameters μ and σ have the following meanings:

- μ is the **mean** of x,
- σ is the **standard deviation** of x, and
- σ^2 is the **variance** of x.

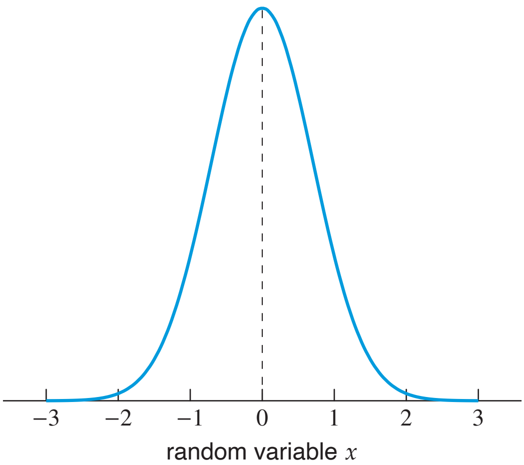

Figure 3.8. PDF for Gaussian random variable x, mean $\mu = 0$, and standard deviation $\sigma = 1/\sqrt{2}$.

Consider the "bell-shaped" Gaussian PDF in figure 3.8. It is always symmetric. The mean μ is its central value and the standard deviation σ is directly related to its width. We will continue to explore the Gaussian distribution in the following lectures, especially in section 4.3.

3.7 Expectation

JH

Recall that a random variable is a function $X : \Omega \to \mathbb{R}$ that maps from the sample space to the reals. Random variables are the arguments of probability mass functions (PMFs) and probability density functions (PDFs).

The **expected value** (or **expectation**) of a random variable is akin to its "average value" and depends on its PMF or PDF. The expected value of a random variable X is denoted $\langle X \rangle$ or $\mathrm{E}[X]$. There are two definitions of the expectation, one for a discrete random variable, the other for a continuous random variable. Before we define, them, however, it is useful to predefine the most fundamental property of a random variable, its **mean**.

Definition 3.1

The *mean* of a random variable X is defined as

$$m_X = \mathrm{E}[X] \,.$$

Let's begin with a discrete random variable.