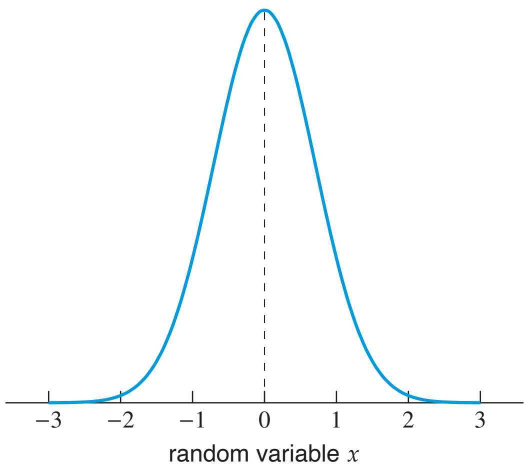

Figure 3.8. PDF for Gaussian random variable x, mean $\mu = 0$, and standard deviation $\sigma = 1/\sqrt{2}$.

Consider the "bell-shaped" Gaussian PDF in figure 3.8. It is always symmetric. The mean μ is its central value and the standard deviation σ is directly related to its width. We will continue to explore the Gaussian distribution in the following lectures, especially in section 4.3.

3.7 Expectation

JH

Recall that a random variable is a function $X : \Omega \to \mathbb{R}$ that maps from the sample space to the reals. Random variables are the arguments of probability mass functions (PMFs) and probability density functions (PDFs).

The **expected value** (or **expectation**) of a random variable is akin to its "average value" and depends on its PMF or PDF. The expected value of a random variable X is denoted $\langle X \rangle$ or $\mathrm{E}[X]$. There are two definitions of the expectation, one for a discrete random variable, the other for a continuous random variable. Before we define, them, however, it is useful to predefine the most fundamental property of a random variable, its **mean**.

Definition 3.1

The *mean* of a random variable X is defined as

$$m_X = \mathrm{E}[X].$$

Let's begin with a discrete random variable.

Definition 3.2

Let K be a discrete random variable and f its PMF. The *expected value* of K is defined as

$$\mathrm{E}[K] = \sum_{\forall k} k f(k).$$

Example 3.8

Given a discrete random variable K with PMF shown below, what is its mean m_K?

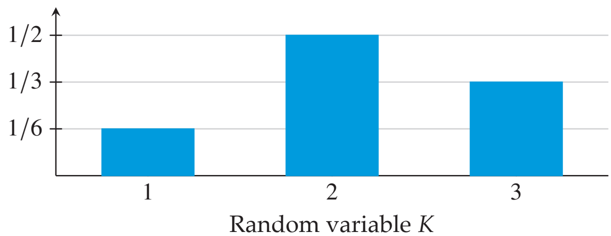

Figure 3.9. PMF of discrete random variable K.

Compute from the definitions:

$$\begin{aligned}\mu_K &= \mathrm{E}[K] \\ &= \sum_{i=1}^{3} k_i f(k_i) \\ &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{3}{6} + 3 \cdot \frac{2}{6} \\ &= \frac{13}{6}.\end{aligned}$$

Let us now turn to the expectation of a continuous random variable.

Definition 3.3

Let X be a continuous random variable and f its PDF. The *expected value* of X is defined as

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x) dx.$$

Example 3.9

Given a continuous random variable X with Gaussian PDF f, what is the expected value of X?

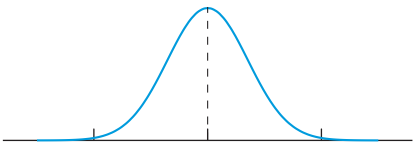

Figure 3.10. Gaussian PDF for random variable X.

Compute from the definition:

$$\begin{aligned} \mathrm{E}[X] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{-(x-\mu)^2}{2\sigma^2} dx. \end{aligned}$$

Substitute $z = x - \mu$:

$$\begin{aligned} \mathrm{E}[X] &= \int_{-\infty}^{\infty} (z+\mu) \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{-z^2}{2\sigma^2} dz \\ &= \mu \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{-z^2}{2\sigma^2} dz + \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z \exp \frac{-z^2}{2\sigma^2} dz. \end{aligned}$$

The first integrand is a Gaussian PDF with its $\mu = 0$, so, by definition, the first integral is 1. The second integrand is an *odd* function, so its improper integral over all z is 0. This leaves

$$\mathrm{E}[X] = \mu.$$

Due to its sum or integral form, the expected value $\mathrm{E}[\cdot]$ has some familiar properties for random variables X and Y and reals a and b.

$$\mathrm{E}[a] = a \tag{3.3}$$

$$\mathrm{E}[X + a] = \mathrm{E}[X] + a \tag{3.4}$$

$$\mathrm{E}[aX] = a\,\mathrm{E}[X] \tag{3.5}$$

$$\mathrm{E}[\mathrm{E}[X]] = \mathrm{E}[X] \tag{3.6}$$

$$\mathrm{E}[aX + bY] = a\,\mathrm{E}[X] + b\,\mathrm{E}[Y]. \tag{3.7}$$

3.8 Central Moments

F2

Given a probability mass function (PMF) or probability density function (PDF) of a random variable, several useful parameters of the random variable can be computed. These are called **central moments**, which quantify parameters relative to its mean.

Definition 3.4

The nth central moment of random variable X, with PDF f, is defined as

$$\mathrm{E}\left[(X-\mu_X)^n\right] = \int_{-\infty}^{\infty} (x-\mu_X)^n f(x) dx.$$

For discrete random variable K with PMF f,

$$\mathrm{E}\left[(K-\mu_K)^n\right] = \sum_{\forall k}^{\infty} (k-\mu_K)^n f(k).$$

Example 3.10

Prove that the first moment of continuous random variable X is zero.

From the definition of the first moment:

$$\begin{aligned}
\mathrm{E}\left[(X-\mu_X)^1\right] &= \int_{-\infty}^{\infty} (x-\mu_X)^1 f(x) dx && (3.8)\\
&= \int_{-\infty}^{\infty} x f(x) dx - \mu_X \int_{-\infty}^{\infty} f(x) dx && \text{(split)}\\
&= \mu_X - \mu_X \cdot 1 && \text{(defs. of } \mu_X \text{ and PDF)}\\
&= 0. && (3.9)
\end{aligned}$$

The second central moment of random variable X is called the **variance** and is denoted

$$\sigma_X^2 \quad \text{or} \quad \mathrm{Var}[X] \quad \text{or} \quad \mathrm{E}\left[(X-\mu_X)^2\right].$$

The variance is a measure of the *width* or *spread* of the PMF or PDF. We usually compute the variance with the formula

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mu_X^2.$$