3.8 Central Moments

F2

Given a probability mass function (PMF) or probability density function (PDF) of a random variable, several useful parameters of the random variable can be computed. These are called **central moments**, which quantify parameters relative to its mean.

Definition 3.4

The nth central moment of random variable X, with PDF f, is defined as

$$E[(X-\mu_X)^n] = \int_{-\infty}^{\infty} (x-\mu_X)^n f(x)dx.$$

For discrete random variable K with PMF f,

$$E[(K-\mu_K)^n] = \sum_{\forall k}^{\infty} (k-\mu_K)^n f(k).$$

Example 3.10

Prove that the first moment of continuous random variable X is zero.

From the definition of the first moment:

$$\begin{aligned} E\left[(X-\mu_X)^1\right] &= \int_{-\infty}^{\infty} (x-\mu_X)^1 f(x)dx && (3.8) \\ &= \int_{-\infty}^{\infty} x f(x)dx - \mu_X \int_{-\infty}^{\infty} f(x)dx && \text{(split)} \\ &= \mu_X - \mu_X \cdot 1 && \text{(defs. of } \mu_X \text{ and PDF)} \\ &= 0. && (3.9) \end{aligned}$$

The second central moment of random variable X is called the **variance** and is denoted

$$\sigma_X^2 \quad \text{or} \quad \mathrm{Var}[X] \quad \text{or} \quad E\left[(X-\mu_X)^2\right].$$

The variance is a measure of the *width* or *spread* of the PMF or PDF. We usually compute the variance with the formula

$$\mathrm{Var}[X] = E\left[X^2\right] - \mu_X^2.$$

Other properties of variance include, for real constant c,

$$\begin{aligned} \operatorname{Var}[c] &= 0 \\ \operatorname{Var}[X+c] &= \operatorname{Var}[X] \\ \operatorname{Var}[cX] &= c^2 \operatorname{Var}[X]. \end{aligned}$$

The **standard deviation** is defined as

$$\sigma_X = \sqrt{\sigma_X^2}.$$

Although the variance is mathematically more convenient, the standard deviation has the same physical units as X, so it is often the more physically meaningful quantity. Due to its meaning as the width or spread of the probability distribution, and its sharing of physical units, it is a convenient choice for error bars on plots of a random variable.

The **skewness** $\operatorname{Skew}[X]$ is a normalized third central moment:

$$\operatorname{Skew}[X] = \frac{\operatorname{E}\left[(X-\mu_X)^3\right]}{\sigma_X^3}.$$

Skewness is a measure of **asymmetry** of a random variable's PDF or PMF. For a symmetric PMF or PDF, such as the Gaussian PDF, $\operatorname{Skew}[X] = 0$.

The **kurtosis** $\operatorname{Kurt}[X]$ is a normalized fourth central moment:

$$\operatorname{Kurt}[X] = \frac{\operatorname{E}\left[(X-\mu_X)^4\right]}{\sigma_X^4}.$$

Kurtosis is a measure of the **tailedness** of a random variable's PDF or PMF. "Heavier" tails yield higher kurtosis.

A Gaussian random variable has PDF with kurtosis 3. Given that for Gaussians both skewness and kurtosis have nice values (0 and 3), we can think of skewness and and kurtosis as measures of similarity to the Gaussian PDF.

3.9 Transforming Random Variables

SR

TODO: describe the theory and formulae

For random variables X and Y with PDFs f_X and f_Y, and with invertible transformation $Y = g(X)$, we have the linear approximation

$$f_Y(y) = \frac{1}{|dy/dx|} f_X(x)\bigg|_{x \mapsto g^{-1}(y)}. \tag{3.10}$$