

4.6 Problems



**Problem 4.1** You need to know the duration of time a certain stage of a brewing process takes. You set up an automated test environment that repeats the test 100 times, recorded in the following JSON<sup>1</sup> data file: <https://math.ricopic.one/bt>. Perform the following analysis.

- a. Download and parse the JSON file (it contains a single array).
- b. Estimate the duration of the process from the sample.
- c. Choose and justify an assumed probability density function for the random variable duration.
- d. Use this PDF model to compute a 99 percent confidence interval for your duration estimate.
- e. Compute your duration confidence interval for the range of confidence values [85, 99.99] percent.<sup>2</sup>
- f. Plot the confidence intervals over the range of confidence in said intervals.

**Problem 4.2** Use linear regression techniques to find the values of  $a$ ,  $b$ ,  $c$ , and  $d$ , in a cubic function of the form,

$$f(x) = ax^3 + bx^2 + cx + d,$$

using the data below.

$x$	$f(x)$
-2.0	-4.7
-1.5	-1.9
-1.0	1.5
-0.5	1.5
0.0	1.4
0.6	0.3
1.1	-1.5
1.6	0.0
2.1	0.6
2.6	4.2

1. JSON is a simple and common programming language-independent data format. For parsing it with Matlab, see `jsondecode` here: <https://math.ricopic.one/75>. For parsing it with Python, see the module `json` here: <https://math.ricopic.one/jb>.

2. Consider using a  $z$ - or  $t$ -score inverse CDF lookup function like `t.ppf` from `scipy.stats`.

**Problem 4.3** 🤖ROBOTIZATION Use linear regression techniques to find the value of  $\tau$  in the function,

$$f(t) = 1 - e^{-\frac{t^2}{\tau}}$$

Using the data below.


$t$	$f(t)$
0.1	0.02
0.6	0.34
1.1	0.74
1.6	0.94
2.1	0.98

**Problem 4.4** 🧑TIRED There are 7 students enrolled in MME 502. If every week 5 students comes to class, for how many weeks could a unique set of 5 students come to class?

**Problem 4.5** 🤪STRANGE Use linear regression techniques to find  $\omega$  and  $\phi$  in the function,  $f(t) = \sin(\omega t + \phi)$  using the data below.

$t$	$f(t)$
0.0	0.53
0.1	0.73
0.2	0.91
0.5	0.92
0.6	0.83
0.7	0.65
0.8	0.42
0.9	0.15
1.0	-0.1
1.1	-0.35
1.2	-0.58
1.3	-0.82
1.4	-0.91
1.7	-0.86
1.8	-0.76
1.9	-0.54

Note: there are an infinite number of solutions to the inverse sine function,  $\sin^{-1}(x) = \pm y + n\pi$  where  $n \in \mathbb{Z}$ . You will have to utilize this definition to get your data in a linear form for fitting.

**Problem 4.6**  The steady-state temperature  $T$  of steam at the outlet of a pipe was measured with a probe. The number of samples  $M$  was 20 and the size  $N$  of each sample was 100. The data can be downloaded at [ricopic.one/mathematical\\_foundations/source/dedicated.json](https://ricopic.one/mathematical_foundations/source/dedicated.json). Download the data and put it in your working directory for analysis.

Estimate the sample mean  $\bar{T}$ , sample variance  $S_T$ , and a 99 percent confidence interval for your estimation of the mean. The data is a list of lists of dimension  $N \times M$  (100-by-20). If you are using Python, load the data with

```
import numpy as np
import json
f = open('dedicated.json',)
Tdata = np.array(json.load(f))
print(f'data excerpt:\n{Tdata[0:3,0:4]}')

data excerpt:
[[241.97683415 213.94281418 220.42666213 225.29771168]
 [241.45909599 222.00636196 214.96718074 246.37793887]
 [230.92953215 217.66017678 227.38480454 210.21311645]]
```



# 5 Vector Calculus



A great many physical situations of interest to engineers can be described by **calculus**. It can describe how quantities continuously change over (say) time and gives tools for computing other quantities. We assume familiarity with the fundamentals of calculus: **limit**, **series**, **derivative**, and **integral**. From these and a basic grasp of vectors, we will outline some of the highlights of **vector calculus**. Vector calculus is particularly useful for describing the physics of, for instance, the following.

**mechanics of particles** wherein is studied the motion of particles and the forcing causes thereof

**rigid-body mechanics** wherein is studied the motion, rotational and translational, and its forcing causes, of bodies considered rigid (undeformable)

**solid mechanics** wherein is studied the motion and deformation, and their forcing causes, of continuous solid bodies (those that retain a specific resting shape)

**fluid mechanics** wherein is studied the motion and its forcing causes of fluids (liquids, gases, plasmas)

**heat transfer** wherein is studied the movement of thermal energy through and among bodies

**electromagnetism** wherein is studied the motion and its forcing causes of electrically charged particles

This last example was in fact very influential in the original development of both vector calculus and **complex analysis**.<sup>1</sup> It is not an exaggeration to say that the topics above comprise the majority of physical topics of interest in engineering.

A good introduction to vector calculus is given by (Kreyszig 2011; Chapters 9–10). Perhaps the most famous and enjoyable treatment is given by (Schey 2005) in the adorably titled *Div, Grad, Curl and All that*.

It is important to note that in much of what follows, we will describe (typically the three-dimensional space of our lived experience) as a **euclidean vector space**:

1. For an introduction to complex analysis, see (Kreyszig 2011; Part D).

an  $n$ -dimensional vector space isomorphic to  $\mathbb{R}^n$ . As we know from linear algebra, any vector  $v \in \mathbb{R}^n$  can be expressed in any number of **bases**. That is, the vector  $v$  is a basis-free object with multiple basis representations. The **components** and **basis vectors** of a vector change with basis changes, but the vector itself is **invariant**. A **coordinate system** is in fact *just a basis*. We are most familiar, of course, with **Cartesian coordinates**, which is the specific orthonormal basis  $\mathbf{b}$  for  $\mathbb{R}^n$ :

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \mathbf{b}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

**Manifolds** are spaces that appear locally as  $\mathbb{R}^n$ , but can be globally rather different and can describe **non-euclidean geometry** wherein euclidean geometry's **parallel postulate** is invalid. Calculus on manifolds is the focus of **differential geometry**, a subset of which we can consider our current study. A motivation for further study of differential geometry is that it is very convenient when dealing with advanced applications of mechanics, such as rigid-body mechanics of robots and vehicles. A very nice mathematical introduction is given by (Lee 2012) and (Bullo and Lewis 2005) give a compact presentation in the context of robotics.

Vector fields have several important properties of interest we'll explore in this chapter. Our goal is to gain an intuition of these properties and be able to perform basic calculation.

## 5.1 Divergence, Surface Integrals, and Flux

### 5.1.1 Flux and Surface Integrals

Consider a surface  $S$ . Let  $\mathbf{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$  be a parametric position vector on a Euclidean vector space  $\mathbb{R}^3$ . Furthermore, let  $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a vector-valued function of  $\mathbf{r}$  and let  $\mathbf{n}$  be a unit-normal vector on a surface  $S$ . The **surface integral**

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS \tag{5.1}$$

which integrates the normal of  $\mathbf{F}$  over the surface. We call this quantity the **flux** of  $\mathbf{F}$  out of the surface  $S$ . This terminology comes from fluid flow, for which the flux is the mass flow rate out of  $S$ . In general, the flux is a measure of a quantity (or field) passing through a surface. For more on computing surface integrals, see Schey (2005; pp. 21-30) and Kreyszig (2011; § 10.6).

