an n-dimensional vector space isomorphic to $\mathbb{R}^n$. As we know from linear algebra, any vector $\boldsymbol{v} \in \mathbb{R}^n$ can be expressed in any number of **bases**. That is, the vector $\boldsymbol{v}$ is a basis-free object with multiple basis representations. The **components** and **basis vectors** of a vector change with basis changes, but the vector itself is **invariant**. A **coordinate system** is in fact *just a basis*. We are most familiar, of course, with **Cartesian coordinates**, which is the specific orthonormal basis $\boldsymbol{b}$ for $\mathbb{R}^n$:

$$\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \boldsymbol{b}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Manifolds are spaces that appear locally as $\mathbb{R}^n$, but can be globally rather different and can describe **non-euclidean geometry** wherein euclidean geometry's **parallel postulate** is invalid. Calculus on manifolds is the focus of **differential geometry**, a subset of which we can consider our current study. A motivation for further study of differential geometry is that it is very convenient when dealing with advanced applications of mechanics, such as rigid-body mechanics of robots and vehicles. A very nice mathematical introduction is given by (Lee 2012) and (Bullo and Lewis 2005) give a compact presentation in the context of robotics.

Vector fields have several important properties of interest we'll explore in this chapter. Our goal is to gain an intuition of these properties and be able to perform basic calculation.

5.1 Divergence, Surface Integrals, and Flux

5.1.1 Flux and Surface Integrals

Consider a surface S. Let $\boldsymbol{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ be a parametric position vector on a Euclidean vector space $\mathbb{R}^3$. Furthermore, let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector-valued function of $\boldsymbol{r}$ and let $\boldsymbol{n}$ be a unit-normal vector on a surface S. The **surface integral**

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S \tag{5.1}$$

which integrates the normal of $\boldsymbol{F}$ over the surface. We call this quantity the **flux** of $\boldsymbol{F}$ out of the surface S. This terminology comes from fluid flow, for which the flux is the mass flow rate out of S. In general, the flux is a measure of a quantity (or field) passing through a surface. For more on computing surface integrals, see Schey (2005; pp. 21-30) and Kreyszig (2011; § 10.6).

5.1.2 Continuity

Consider the flux out of a surface S that encloses a volume ΔV, divided by that volume:

$$\frac{1}{\Delta V}\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{5.2}$$

This gives a measure of flux per unit volume for a volume of space. Consider its physical meaning when we interpret this as fluid flow: all fluid that enters the volume is negative flux and all that leaves is positive. If physical conditions are such that we expect no fluid to enter or exit the volume via what is called a **source** or a **sink**, and if we assume the density of the fluid is uniform (this is called an **incompressible** fluid), then all the fluid that enters the volume must exit and we get

$$\frac{1}{\Delta V}\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S = 0. \tag{5.3}$$

This is called a **continuity equation**, although typically this name is given to equations of the form in the next section. This equation is one of the governing equations in continuum mechanics.

5.1.3 Divergence

Let's take the flux-per-volume as the volume $\Delta V \to 0$ we obtain the following.

Equation 5.4 divergence: integral form

$$\lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}S.$$

This is called the **divergence** of $\boldsymbol{F}$ and is defined at each point in $\mathbb{R}^3$ by taking the volume to zero about it. It is given the shorthand $\operatorname{div}\boldsymbol{F}$.

What interpretation can we give this quantity? It is a measure of the vector field's flux outward through a surface containing an infinitesimal volume. When we consider a fluid, a positive divergence is a local decrease in density and a negative divergence is a density increase. If the fluid is incompressible and has no sources or sinks, we can write the continuity equation

$$\operatorname{div}\boldsymbol{F} = 0. \tag{5.5}$$

In the Cartesian basis, it can be shown that the divergence is easily computed from the field

$$\boldsymbol{F} = F_x \hat{\boldsymbol{i}} + F_y \hat{\boldsymbol{j}} + F_z \hat{\boldsymbol{k}} \tag{5.6}$$

as follows.

Equation 5.7 divergence: differential form

$$\operatorname{div} \boldsymbol{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$$

5.1.4 Exploring Divergence

Divergence is perhaps best explored by considering it for a vector field in $\mathbb{R}^2$. Such a field $\boldsymbol{F} = F_x \hat{\boldsymbol{i}} + F_y \hat{\boldsymbol{j}}$ can be represented as a "quiver" plot. If we overlay the quiver plot over a "color density" plot representing $\operatorname{div} \boldsymbol{F}$, we can increase our intuition about the divergence.

First, load some Python packages.

```
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt
from matplotlib.ticker import LogLocator
from matplotlib.colors import *
from sympy.utilities.lambdify import lambdify
```

Now we define some symbolic variables and functions.

```
x = sp.Symbol('x', real=True)
y = sp.Symbol('y', real=True)
F_x = sp.Function('F_x')(x, y)
F_y = sp.Function('F_y')(x, y)
```

Rather than repeat code, let's write a single function `quiver_plotter_2D()` to make several of these plots.

```
def quiver_plotter_2D(
        field={},
        grid_width=3, grid_decimate_x=8, grid_decimate_y=8,
        norm=Normalize(), density_operation='div',
        print_density=True):
    x, y = sp.symbols('x y', real=True)
    F_x, F_y = sp.Function('F_x')(x, y), sp.Function('F_y')(x, y)
    field_sub = field
    # Calculate density
    den = F_x.diff(x) + F_y.diff(y) if density_operation == 'div' else None
    if den is None:
```

```
        raise ValueError(f'Unknown density operation: {density_operation}')
    den_simp = den.subs(field_sub).doit().simplify()
    if den_simp.is_constant():
        print('Warning: density operator is constant (no density plot)')
    if print_density:
        print(f'The {density_operation} is:')
        print(den_simp)
    # Lambdify for numerics
    F_x_sub = F_x.subs(field_sub)
    F_y_sub = F_y.subs(field_sub)
    F_x_fun = sp.lambdify((x, y), F_x.subs(field_sub), 'numpy')
    F_y_fun = sp.lambdify((x, y), F_y.subs(field_sub), 'numpy')
    if F_x_sub.is_constant:
        F_x_fun1 = F_x_fun  # Dummy
        F_x_fun = lambda x, y: F_x_fun1(x, y) * np.ones(x.shape)
    if F_y_sub.is_constant:
        F_y_fun1 = F_y_fun  # Dummy
        F_y_fun = lambda x, y: F_y_fun1(x, y) * np.ones(x.shape)
    if not den_simp.is_constant():
        den_fun = sp.lambdify((x, y), den_simp, 'numpy')
    # Create grid
    w = grid_width
    Y, X = np.mgrid[-w:w:100j, -w:w:100j]
    # Evaluate numerically
    F_x_num = F_x_fun(X, Y)
    F_y_num = F_y_fun(X, Y)
    if not den_simp.is_constant():
        den_num = den_fun(X, Y)
    # Plot
    fig, ax = plt.subplots()
    if not den_simp.is_constant():
        cmap = plt.get_cmap('coolwarm')
        im = plt.pcolormesh(X, Y, den_num, cmap=cmap, norm=norm)
        plt.colorbar()
    dx, dy = grid_decimate_y, grid_decimate_x
    plt.quiver(X[::dx, ::dy], Y[::dx, ::dy], F_x_num[::dx, ::dy],
               F_y_num[::dx, ::dy], units='xy', scale=10)
    plt.title(fr'$F(x, y) = \left[{sp.latex(F_x.subs(field_sub))},' +
              fr'{sp.latex(F_y.subs(field_sub))}\right]$')
    return fig, ax
```

Let's inspect several cases.

```
fig, ax = quiver_plotter_2D(field={F_x: x**2, F_y: y**2})
plt.draw()
```

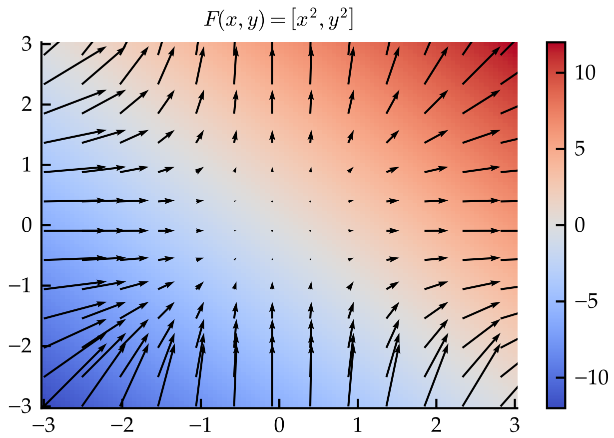

Figure 5.1. Quiver plot of $F(x, y) = \left[x^2, y^2\right]$

```
fig, ax = quiver_plotter_2D(field={F_x: x*y, F_y: x*y})
plt.draw()
```

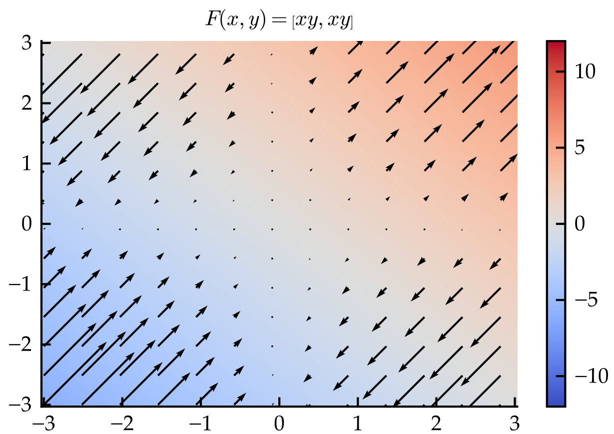

Figure 5.2. Quiver plot of $F(x, y) = [xy, xy]$

```
fig, ax = quiver_plotter_2D(field={F_x: x**2 + y**2, F_y: x**2 + y**2})
plt.draw()
```

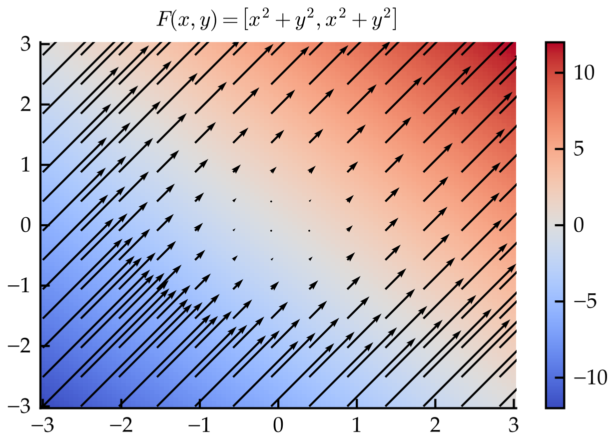

Figure 5.3. Quiver plot of $F(x,y) = \left[x^2 + y^2, x^2 + y^2\right]$

```
fig, ax = quiver_plotter_2D(
  field={F_x: x**2/sp.sqrt(x**2+y**2), F_y: y**2/sp.sqrt(x**2+y**2)},
  norm=SymLogNorm(linthresh=.3, linscale=.3)
)
plt.show()
```

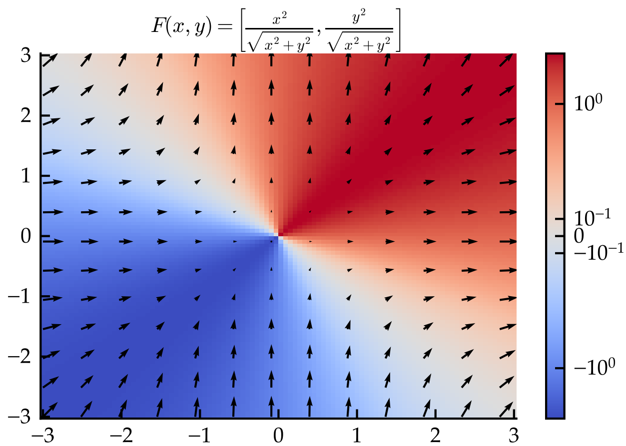

Figure 5.4. Quiver plot of $F(x,y) = \left[x^2/\sqrt{x^2 + y^2}, y^2/\sqrt{x^2 + y^2}\right]$

5.2 Curl, Line Integrals, and Circulation

5.2.1 Line Integrals

Consider a curve C in a Euclidean vector space $\mathbb{R}^3$. Let $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ be a parametric representation of C. Furthermore, let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector-valued function of $\boldsymbol{r}$ and let $\boldsymbol{r}'(t)$ be the tangent vector. The **line integral** is

$$\int\limits_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, \mathrm{d}t \tag{5.8}$$

which integrates $\boldsymbol{F}$ along the curve. For more on computing line integrals, see (Schey 2005; pp. 63-74) and (Kreyszig 2011; § 10.1 and 10.2).

If $\boldsymbol{F}$ is a **force** being applied to an object moving along the curve C, the line integral is the **work** done by the force. More generally, the line integral integrates $\boldsymbol{F}$ along the tangent of C.

5.2.2 Circulation

Consider the line integral over a closed curve C, denoted by

$$\oint\limits_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, \mathrm{d}t. \tag{5.9}$$

We call this quantity the **circulation** of $\boldsymbol{F}$ around C.

For certain vector-valued functions $\boldsymbol{F}$, the circulation is zero for every curve. In these cases (static electric fields, for instance), this is sometimes called the **the law of circulation**.

5.2.3 Curl

Consider the division of the circulation around a curve in $\mathbb{R}^3$ by the surface area it encloses ΔS,

$$\frac{1}{\Delta S} \oint\limits_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, \mathrm{d}t. \tag{5.10}$$

In a manner analogous to the operation that gaves us the divergence, let's consider shrinking this curve to a point and the surface area to zero,

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint\limits_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, \mathrm{d}t. \tag{5.11}$$

We call this quantity the "scalar" **curl** of $\boldsymbol{F}$ at each point in $\mathbb{R}^3$ *in the direction normal to* ΔS as it shrinks to zero. Taking three (or n for $\mathbb{R}^n$) "scalar" curls in indepedent