5.3 Gradient

5.3.1 Gradient

The **gradient** grad of a scalar-valued function $f : \mathbb{R}^3 \to \mathbb{R}$ is a vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$; that is, grad f is a vector-valued function on $\mathbb{R}^3$. The gradient's local **direction** and **magnitude** are those of the local maximum rate of increase of f. This makes it useful in optimization (e.g., in the method of gradient descent).

This principle tells us that nature's laws quite frequently seem to be derivable by assuming a certain quantity—called *action*—is minimized. Considering, then, that the gradient supplies us with a tool for optimizing functions, it is unsurprising that the gradient enters into the equations of motion of many physical quantities.

The gradient is coordinate-independent, but its coordinate-free definitions don't add much to our intuition.

Equation 5.19 gradient: cartesian coordinates

$$\operatorname{grad} f = \begin{bmatrix} \partial_x f & \partial_y f & \partial_z f \end{bmatrix}^\top$$

5.3.2 Vector Fields from Gradients Are Special

Although all gradients are vector fields, not all vector fields are gradients. That is, given a vector field $\boldsymbol{F}$, it may or may not be equal to the gradient of any scalar-valued function f. Let's say of a vector field that is a gradient that it has **gradience**.[3] Those vector fields that *are* gradients have special properties. Surprisingly, those properties are connected to path independence and curl. It can be shown that iff a field is a gradient, line integrals of the field are path independent. That is, for a vector field,

$$\text{gradience} \Leftrightarrow \text{path independence.} \tag{5.20}$$

Considering what we know from section 5.2 about path independence we can expand figure 5.5 to obtain figure 5.10.

3. This is nonstandard terminology, but we're bold.

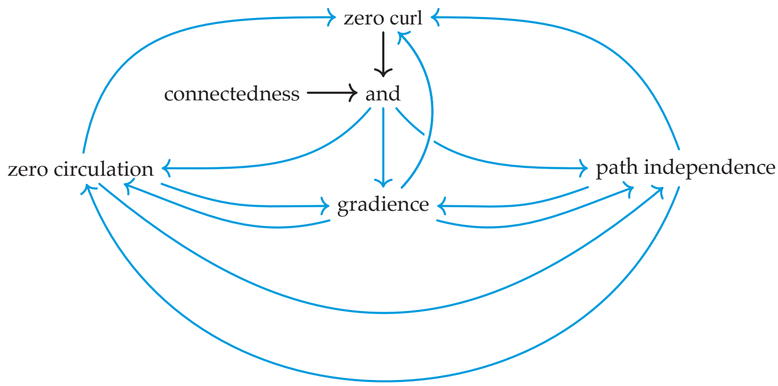

Figure 5.10. An implication graph relating gradience, zero curl, zero circulation, path independence, and connectedness. Green edges represent implication (a implies b) and black edges represent logical conjunctions.

One implication is that *gradients have zero curl*! Many important fields that describe physical interactions (e.g., static electric fields, Newtonian gravitational fields) are gradients of scalar fields called **potentials**.

5.3.3 Exploring Gradient

Gradient is perhaps best explored by considering it for a scalar field on $\mathbb{R}^2$. Such a field in cartesian coordinates $f(x, y)$ has gradient

$$\operatorname{grad} f = \begin{bmatrix} \partial_x f & \partial_y f \end{bmatrix}^\top \tag{5.21}$$

That is, $\operatorname{grad} f = \boldsymbol{F} = \partial_x f \ \hat{\boldsymbol{\imath}} + \partial_y f \ \hat{\boldsymbol{\jmath}}$. If we overlay a quiver plot of $\boldsymbol{F}$ over a "color density" plot representing the f, we can increase our intuition about the gradient.

First, load some Python packages.

```
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt
from matplotlib.ticker import LogLocator
from matplotlib.colors import *
```

Now we define some symbolic variables and functions.

```
x, y = sp.symbols('x y', real=True)
```

Rather than repeat code, let's write a single function `grad_plotter_2D()` to make several of these plots.

```python
def grad_plotter_2D(
  field=x*y, grid_width=3, grid_decimate_x=8, grid_decimate_y=8,
  norm=None, # Density plot normalization
  scale=None, # Arrow length scale (auto)
  print_vector=True, mask=False, # Mask vector lengths
):
  # Define symbolics
  x, y = sp.symbols('x y', real=True)
  field = sp.sympify(field)
  # Compute vector field
  F_x = field.diff(x).simplify()
  F_y = field.diff(y).simplify()
  if field.is_constant():
    print('Warning: field is constant (no plot)')
  if print_vector:
    print(f'The gradient is:')
    print(sp.Array([F_x, F_y]))
  # Lambdify for numerics
  F_x_fun = sp.lambdify((x, y), F_x, 'numpy')
  F_y_fun = sp.lambdify((x, y), F_y, 'numpy')
  if F_x.is_constant:
    F_x_fun1 = F_x_fun # Dummy
    F_x_fun = lambda x, y: F_x_fun1(x, y) * np.ones(x.shape)
  if F_y.is_constant:
    F_y_fun1 = F_y_fun # Dummy
    F_y_fun = lambda x, y: F_y_fun1(x, y) * np.ones(x.shape)
  if not field.is_constant():
    den_fun = sp.lambdify((x, y), field, 'numpy')
  # Create grid
  w = grid_width
  Y, X = np.mgrid[-w:w:100j, -w:w:100j]
  # Evaluate numerically
  F_x_num = F_x_fun(X, Y)
  F_y_num = F_y_fun(X, Y)
  if not field.is_constant():
    den_num = den_fun(X, Y)
  # Mask F_x and F_y
  if mask:
    masking_a = np.sqrt(np.square(F_x_num) + np.square(F_y_num))
    F_x_num = np.ma.masked_where(masking_a > w / 5., F_x_num)
    F_y_num = np.ma.masked_where(masking_a > w / 5., F_y_num)
  # Plot
  if not field.is_constant():
    fig, ax = plt.subplots()
    cmap = plt.get_cmap('coolwarm')
    im = plt.pcolormesh(X, Y, den_num, cmap=cmap, norm=norm)
    plt.colorbar()
```

```
    dx = grid_decimate_y
    dy = grid_decimate_x
    plt.quiver(
      X[::dx, ::dy], Y[::dx, ::dy],
      F_x_num[::dx, ::dy], F_y_num[::dx, ::dy],
      units='xy', scale=scale
    )
    plt.title(f'$f(x,y) = {sp.latex(field)}$')
    return fig, ax
  return 1, 1
```

Let's inspect several cases. While considering the following plots, remember that they all have zero curl!

```
fig, ax = grad_plotter_2D(field=x)
plt.draw()
```

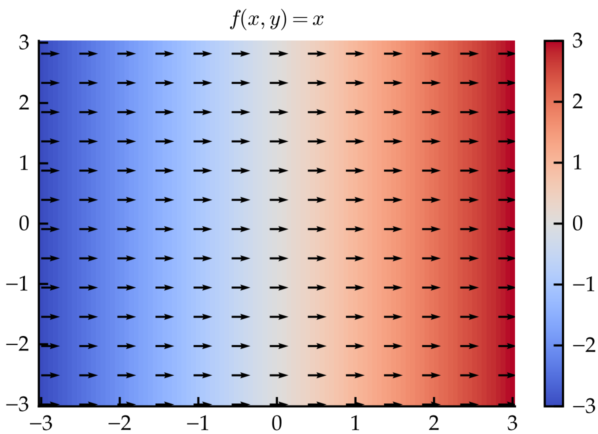

Figure 5.11. Gradient of $f(x,y) = x$

```
fig, ax = grad_plotter_2D(field=x+y)
plt.draw()
```

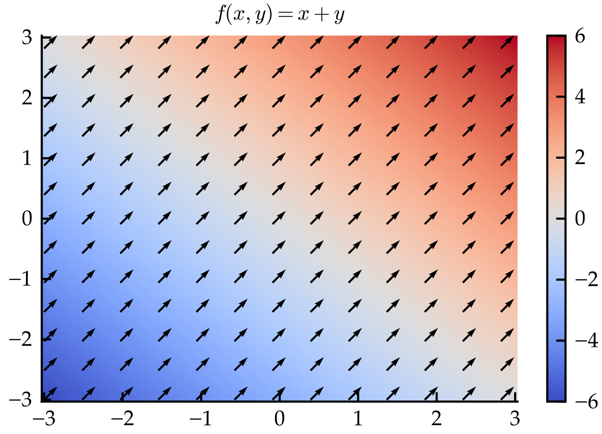

Figure 5.12. Gradient of $f(x, y) = x + y$

```
fig, ax = grad_plotter_2D(field=1)
```

5.3.3.1 Gravitational Potential Gravitational potentials have the form of 1/distance. Let's check out the gradient.

```
fig, ax = grad_plotter_2D(
  field=1/sp.sqrt(x**2+y**2),
  norm=SymLogNorm(linthresh=.3, linscale=.3), mask=True,
)
plt.draw()
```

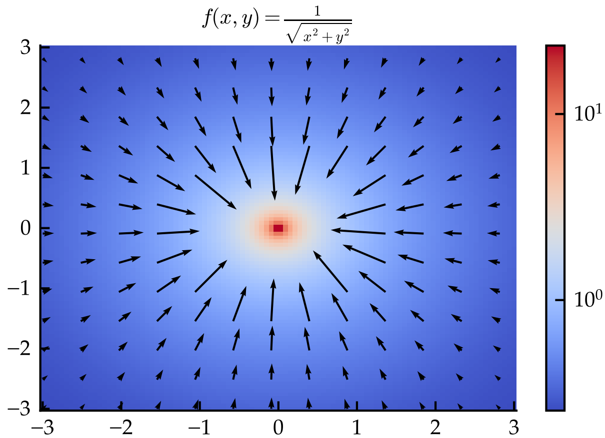

Figure 5.13. Gradient of $f(x,y) = 1/\sqrt{x^2+y^2}$

5.3.3.2 Conic Section Fields Gradients of **conic section** fields can be explored. The following is called a **parabolic field**.

```
fig, ax = grad_plotter_2D(field=x**2)
plt.draw()
```

The following are called **elliptic fields**.

```
fig, ax = grad_plotter_2D(field=x**2 + y**2)
plt.draw()
```

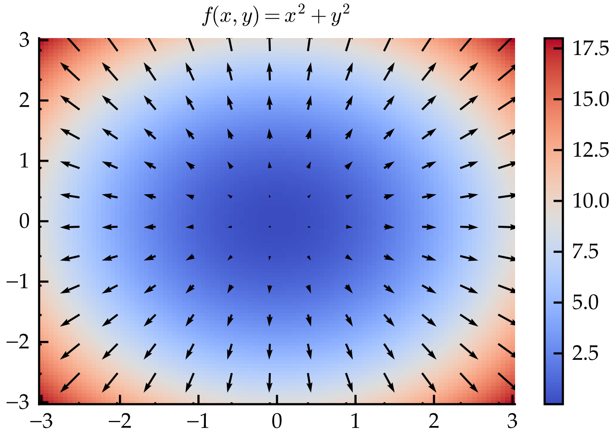

Figure 5.14. Gradient of $f(x, y) = x^2 + y^2$

```
fig, ax = grad_plotter_2D(field=-x**2 - y**2)
plt.draw()
```

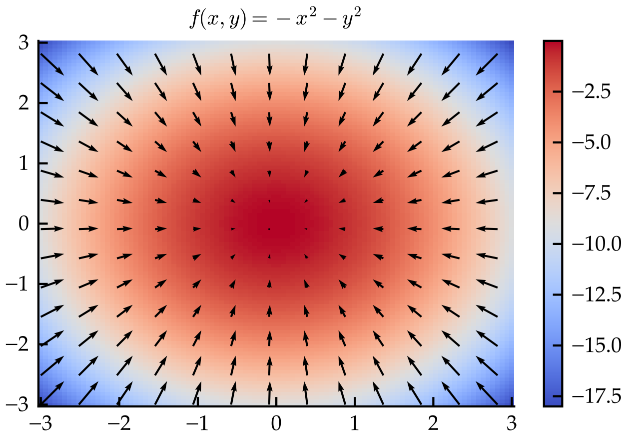

Figure 5.15. Gradient of $f(x, y) = -x^2 - y^2$

The following is called a **hyperbolic field**.

```
fig, ax = grad_plotter_2D(field=x**2 - y**2)
plt.show()
```

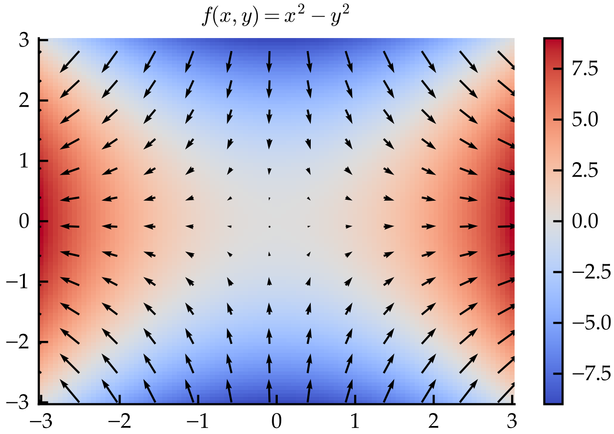

Figure 5.16. Gradient of $f(x, y) = x^2 - y^2$

5.4 Stokes and Divergence Theorems

A4

Two theorems allow us to exchange certain integrals in $\mathbb{R}^3$ for others that are easier to evaluate.

5.4.1 The Divergence Theorem

The **divergence theorem** asserts the equality of the surface integral of a vector field $\boldsymbol{F}$ and the **triple integral** of div $\boldsymbol{F}$ over the volume V enclosed by the surface S in $\mathbb{R}^3$. That is,

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S = \iiint_V \operatorname{div} \boldsymbol{F} \, \mathrm{d}V.$$

Caveats are that V is a closed region bounded by the **orientable**[4] surface S and that $\boldsymbol{F}$ is continuous and continuously differentiable over a region containing V. This theorem makes some intuitive sense: we can think of the divergence inside the volume "accumulating" via the triple integration and equaling the corresponding surface integral. For more on the divergence theorem, see (Kreyszig 2011; § 10.7) and (Schey 2005; pp. 45-52).

A lovely application of the divergence theorem is that, for any quantity of conserved stuff (mass, charge, spin, etc.) distributed in a spatial $\mathbb{R}^3$ with time-dependent density $\rho : \mathbb{R}^4 \to \mathbb{R}$ and velocity field $\boldsymbol{v} : \mathbb{R}^4 \to \mathbb{R}^3$, the divergence theorem can be

4. A surface is orientable if a consistent normal direction can be defined. Most surfaces are orientable, but some, notably the Möbius strip, cannot be. See (Kreyszig 2011; § 10.6) for more.