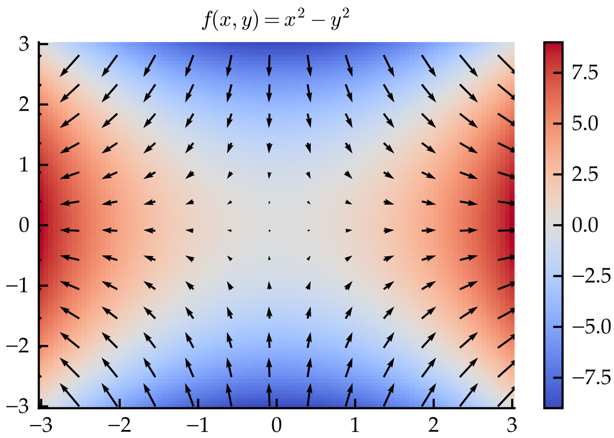

Figure 5.16. Gradient of $f(x, y) = x^2 - y^2$

5.4 Stokes and Divergence Theorems

A4

Two theorems allow us to exchange certain integrals in $\mathbb{R}^3$ for others that are easier to evaluate.

5.4.1 The Divergence Theorem

The **divergence theorem** asserts the equality of the surface integral of a vector field $\boldsymbol{F}$ and the **triple integral** of div $\boldsymbol{F}$ over the volume V enclosed by the surface S in $\mathbb{R}^3$. That is,

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n} \, \mathrm{d}S = \iiint_V \operatorname{div} \boldsymbol{F} \, \mathrm{d}V.$$

Caveats are that V is a closed region bounded by the **orientable**[4] surface S and that $\boldsymbol{F}$ is continuous and continuously differentiable over a region containing V. This theorem makes some intuitive sense: we can think of the divergence inside the volume "accumulating" via the triple integration and equaling the corresponding surface integral. For more on the divergence theorem, see (Kreyszig 2011; § 10.7) and (Schey 2005; pp. 45-52).

A lovely application of the divergence theorem is that, for any quantity of conserved stuff (mass, charge, spin, etc.) distributed in a spatial $\mathbb{R}^3$ with time-dependent density $\rho : \mathbb{R}^4 \to \mathbb{R}$ and velocity field $\boldsymbol{v} : \mathbb{R}^4 \to \mathbb{R}^3$, the divergence theorem can be

4. A surface is orientable if a consistent normal direction can be defined. Most surfaces are orientable, but some, notably the Möbius strip, cannot be. See (Kreyszig 2011; § 10.6) for more.

applied to find that

$$\partial_t \rho = -\operatorname{div}(\rho \boldsymbol{v}),$$

which is a more general form of a **continuity equation**, one of the governing equations of many physical phenomena. For a derivation of this equation, see (pp. 49-52).

5.4.2 The Kelvin-Stokes' Theorem

The **Kelvin-Stokes' theorem** asserts the equality of the circulation of a vector field $\boldsymbol{F}$ over a closed curve C and the surface integral of curl $\boldsymbol{F}$ over a surface S that has boundary C. That is, for $\boldsymbol{r}(t)$ a parameterization of C and surface normal $\boldsymbol{n}$,

$$\oint_C \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\,\mathrm{d}t = \iint_S \boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{F}\,\mathrm{d}S.$$

Caveats are that S is **piecewise smooth**,[5] its boundary C is a piecewise smooth simple closed curve, and $\boldsymbol{F}$ is continuous and continuously differentiable over a region containing S. This theorem is also somewhat intuitive: we can think of the divergence over the surface "accumulating" via the surface integration and equaling the corresponding circulation. For more on the Kelvin-Stokes' theorem, see (Kreyszig 2011; § 10.9) and (Schey 2005; pp. 93-102).

5.4.3 Related Theorems

Greene's theorem is a two-dimensional special case of the Kelvin-Stokes' theorem. It is described by (Kreyszig 2011; § 10.9).

It turns out that all of the above theorems (and the fundamental theorem of calculus, which relates the derivative and integral) are special cases of the **generalized Stokes' theorem** defined by differential geometry. We would need a deeper understanding of differential geometry to understand this theorem. For more, see (Lee 2012; Ch. 16).

5. A surface is *smooth* if its normal is continuous everywhere. It is *piecewise smooth* if it is composed of a finite number of smooth surfaces.

5.5 Problems

DH

Problem 5.1 LIGHT Consider a vector field $\boldsymbol{F}: \mathbb{R}^3 \to \mathbb{R}^3$ defined in Cartesian coordinates (x, y, z) as

$$\boldsymbol{F} = [x^2 - y^2, y^2 - z^2, z^2 - x^2]. \tag{5.22}$$

a. Compute the divergence of $\boldsymbol{F}$.
b. Compute the curl of $\boldsymbol{F}$.
c. Prove that, in a simply connected region of $\boldsymbol{R}^3$, line integrals of $\boldsymbol{F}$ are path-dependent.
d. Prove that $\boldsymbol{F}$ is *not* the gradient of a potential (scalar) function (i.e., that it does not have gradience, as we've called it).

Problem 5.2 HIKE The altitude of (x, y) points on a nearby mountain are modeled on the domain $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ as,

$$f(x, y) = 2 - \frac{x^2}{4} + \cos(\frac{\pi}{2} y).$$

Using this model of the mountain:

a. Find the 3 dimensional path you would travel on if you were to start from the trailhead at $(x, y) = (-1, -1.5)$ and head in a straight line to the top of the mountain at $(0, 0)$.
b. Given the definition of work $W = \int_C F(r) \cdot dr$, write the equation for $F(r)$ from the acceleration of gravity and assuming a mass of 50 kg.
c. Solve for the work to climb the mountain on your path from part **a**.
d. Once you get to the trailhead your friend wants to take a different route that they think will take less work. Prove that it takes the same amount of work, no matter what route you take to the top of the mountain.
e. On your way up the mountain you notice you have altitude sickness at location $(-1, -0.75)$ and need to get to a lower altitude as quickly as possible. What direction should you go to descend the fastest? Write your answer as a vector pointing in the direction you should go.

Problem 5.3 CRABRANGOON Consider a vector field $\boldsymbol{F}: \mathbb{R}^3 \to \mathbb{R}^3$ defined in Cartesian coordinates (x, y, z) as

$$\boldsymbol{F} = \begin{bmatrix} -3x^2 & -3y^2 & 0 \end{bmatrix}^\top. \tag{5.23}$$

a. Compute the divergence of $\boldsymbol{F}$.
b. Compute the curl of $\boldsymbol{F}$.