5.5 Problems

DH

Problem 5.1 LIGHT Consider a vector field $\boldsymbol{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined in Cartesian coordinates (x, y, z) as

$$\boldsymbol{F} = [x^2 - y^2, y^2 - z^2, z^2 - x^2]. \tag{5.22}$$

a. Compute the divergence of $\boldsymbol{F}$.
b. Compute the curl of $\boldsymbol{F}$.
c. Prove that, in a simply connected region of $\boldsymbol{R}^3$, line integrals of $\boldsymbol{F}$ are path-dependent.
d. Prove that $\boldsymbol{F}$ is *not* the gradient of a potential (scalar) function (i.e., that it does not have gradience, as we've called it).

Problem 5.2 HIKE The altitude of (x, y) points on a nearby mountain are modeled on the domain $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ as,

$$f(x, y) = 2 - \frac{x^2}{4} + \cos(\frac{\pi}{2} y).$$

Using this model of the mountain:

a. Find the 3 dimensional path you would travel on if you were to start from the trailhead at $(x, y) = (-1, -1.5)$ and head in a straight line to the top of the mountain at $(0, 0)$.
b. Given the definition of work $W = \int_C F(r) \cdot dr$, write the equation for $F(r)$ from the acceleration of gravity and assuming a mass of 50 kg.
c. Solve for the work to climb the mountain on your path from part **a**.
d. Once you get to the trailhead your friend wants to take a different route that they think will take less work. Prove that it takes the same amount of work, no matter what route you take to the top of the mountain.
e. On your way up the mountain you notice you have altitude sickness at location $(-1, -0.75)$ and need to get to a lower altitude as quickly as possible. What direction should you go to descend the fastest? Write your answer as a vector pointing in the direction you should go.

Problem 5.3 CRABRANGOON Consider a vector field $\boldsymbol{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined in Cartesian coordinates (x, y, z) as

$$\boldsymbol{F} = \begin{bmatrix} -3x^2 & -3y^2 & 0 \end{bmatrix}^\top . \tag{5.23}$$

a. Compute the divergence of $\boldsymbol{F}$.
b. Compute the curl of $\boldsymbol{F}$.

c. Prove that, in a simply connected region of $\boldsymbol{R}^3$, line integrals of $\boldsymbol{F}$ are path-independent.
d. Prove that $\boldsymbol{F}$ is the gradient of a potential (scalar) function (i.e., that it has gradience, as we've called it).
e. Identify a potential function ϕ for which $\boldsymbol{F}$ is the gradient. Is this the only such function?

6 Fourier and Orthogonality

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In this chapter we will explore Fourier series and transforms.

6.1 Fourier Series

OF

Fourier series are mathematical series that can represent a periodic signal as a sum of sinusoids at different amplitudes and frequencies. They are useful for solving for the response of a system to periodic inputs. However, they are probably most important *conceptually*: they are our gateway to thinking of signals in the **frequency domain**—that is, as functions of *frequency* (not time). To represent a function as a Fourier series is to *analyze* it as a sum of sinusoids at different frequencies[1] ω_n and amplitudes a_n. Its **frequency spectrum** is the functional representation of amplitudes a_n versus frequency ω_n.

Let's begin with the definition.

Definition 6.1

The *Fourier analysis* of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$a_0 = \frac{2}{T}\int_T y(t)dt$$

$$a_n = \frac{2}{T}\int_T y(t)\cos(\omega_n t)dt$$

$$b_n = \frac{2}{T}\int_T y(t)\sin(\omega_n t)dt.$$

1. It's important to note that the symbol ω_n, in this context, is not the natural frequency, but a frequency indexed by integer n.