6.3 Generalized Fourier Series and Orthogonality

UJ

Let $f:\mathbb{R}\to\mathbb{C}$, $g:\mathbb{R}\to\mathbb{C}$, and $w:\mathbb{R}\to\mathbb{C}$ be complex functions. For square-integrable[2] f, g, and w, the **inner product** of f and g with **weight function** w over the interval $[a,b]\subseteq\mathbb{R}$ is[3]

$$\langle f,g\rangle_w = \int_a^b f(x)\overline{g}(x)w(x)\,\mathrm{d}x$$

where $\overline{g}$ denotes the complex conjugate of g. The inner product of functions can be considered analogous to the inner (or dot) product of vectors.

The fourier series components can be found by a special property of the sin and cos functions called **orthogonality**. In general, functions f and g from above are *orthogonal* over the interval $[a,b]$ iff

$$\langle f,g\rangle_w = 0$$

for weight function w. Similar to how a set of orthogonal vectors can be a basis for a vector space, a set of orthogonal functions can be a **basis** for a **function space**: a vector space of functions from one set to another (with certain caveats).

In addition to some sets of sinusoids, there are several other important sets of functions that are orthogonal. For instance, sets of **legendre polynomials** (Kreyszig 2011; § 5.2) and **bessel functions** (§ 5.4) are orthogonal.

As with sinusoids, the orthogonality of some sets of functions allows us to compute their series components. Let functions $f_0, f_1, \cdots$ be orthogonal with respect to weight function w on interval $[a,b]$ and let $\alpha_0, \alpha_1, \cdots$ be real constants. A **generalized fourier series** is (§ 11.6)

$$f(x) = \sum_{m=0}^{\infty} \alpha_m f_m(x)$$

and represents a function f as a convergent series. It can be shown that the **Fourier components** α_m can be computed from

$$\alpha_m = \frac{\langle f,f_m\rangle_w}{\langle f_m,f_m\rangle_w}.$$

In keeping with our previous terminology for fourier series, section 6.3 and section 6.3 are called general fourier **synthesis** and **analysis**, respectively.

For the aforementioned legendre and bessel functions, the generalized fourier series are called **fourier-legendre** and **fourier-bessel series** (§ 11.6). These and

2. A function f is square-integrable if $\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x < \infty$.

3. This definition of the inner product can be extended to functions on $\mathbb{R}^2$ and $\mathbb{R}^3$ domains using double- and triple-integration. See (Schey 2005; p. 261).

the standard fourier series (section 6.1) are of particular interest for the solution of partial differential equations (chapter 7).

6.4 Problems

3A

Problem 6.1 STANISLAW Explain, in your own words (supplementary drawings are ok), what the *frequency domain* is, how we derive models in it, and why it is useful.

Problem 6.2 PUG Consider the function

$$f(t) = 8\cos(t) + 6\sin(2t) + \sqrt{5}\cos(4t) + 2\sin(4t) + \cos(6t - \pi/2).$$

(a) Find the (harmonic) magnitude and (harmonic) phase of its Fourier series components. (b) Sketch its magnitude and phase spectra. *Hint: no Fourier integrals are necessary to solve this problem.*

Problem 6.3 PONYO Consider the function with $a > 0$

$$f(t) = e^{-a|t|}.$$

From the transform definition, derive the Fourier transform $F(\omega)$ of $f(t)$. Simplify the result such that it is clear the expression is real (no imaginary component).

Problem 6.4 SEESAW Consider the periodic function $f : \mathbb{R} \to \mathbb{R}$ with period T defined for one period as

$$f(t) = at \quad \text{for } t \in (-T/2, T/2] \tag{6.9}$$

where $a, T \in \mathbb{R}$. Perform a fourier series analysis on f. Letting $a = 5$ and $T = 1$, plot f along with the partial sum of the fourier series synthesis, the first 50 nonzero components, over $t \in [-T, T]$.