important topic. Many PDEs have no known analytic solution, so for these numeric solution is the best available option.[1] However, it is important to note that the insight one can gain from an analytic solution is often much greater than that from a numeric solution. This is easily understood when one considers that a numeric solution is an approximation for a specific set of initial and boundary conditions. Typically, very little can be said of what would happen in general, although this is often what we seek to know. So, despite the importance of numeric solution, one should always prefer an analytic solution.

Three good texts on PDEs for further study are (Kreyszig 2011; Ch. 12), (Strauss 2007), and (Haberman 2018).

7.1 Classifying PDEs

PDEs often have an infinite number of solutions; however, when applying them to physical systems, we usually assume that a deterministic, or at least a probabilistic, sequence of events will occur. Therefore, we impose additonal constraints on a PDE, usually in the form of

1. **initial conditions**, values of independent variables over all space at an initial time and
2. **boundary conditions**, values of independent variables (or their derivatives) over all time.

Ideally, imposing such conditions leaves us with a **well-posed problem**, which has three aspects. (Bove, Colombini, and Santo 2006; § 1.5)

existence There exists at least one solution.
uniqueness There exists at most one solution.
stability If the PDE, boundary conditons, or initial conditions are changed slightly, the solution changes only slightly.

As with ODEs, PDEs can be **linear** or **nonlinear**; that is, the dependent variables and their derivatives can appear in only linear combinations (linear PDE) or in one or more nonlinear combination (nonlinear PDE). As with ODEs, there are more known analytic solutions to linear PDEs than nonlinear PDEs.

The **order** of a PDE is the order of its highest partial derivative. A great many physical models can be described by **second-order PDEs** or systems thereof. Let u be an independent scalar variable, a function of m temporal and spatial variables $x_i \in \mathbb{R}^n$. A second-order linear PDE has the form, for coefficients $\alpha, \beta, \gamma, and \delta$, and

1. There are some analytic techniques for gaining insight into PDEs for which there are no known solutions, such as considering the *phase space*. This is an active area of research; for more, see (Bove, Colombini, and Santo 2006).

real functions of x_i, (Strauss 2007; § 1.6)

$$\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\partial^2_{x_i x_j}u}_{\text{second-order terms}} + \underbrace{\sum_{k=1}^{m}\left(\gamma_k\partial_{x_k}u + \delta_k u\right)}_{\text{first- and zeroth-order terms}} = \underbrace{f(x_1, \cdots, x_n)}_{\text{forcing}}$$

where f is called a **forcing function**. When f is zero, section 7.1 is called **homogeneous**. We can consider the coefficients α_{ij} to be components of a matrix A with rows indexed by i and columns indexed by j. There are four prominent classes defined by the eigenvalues of A:

elliptic the eigenvalues all have the same sign,
parabolic the eigenvalues have the same sign except one that is zero,
hyperbolic exactly one eigenvalue has the opposite sign of the others, and
ultrahyperbolic at least two eigenvalues of each signs.

The first three of these have received extensive treatment. They are named after conic sections due to the similarity the equations have with polynomials when derivatives are considered analogous to powers of polynomial variables. For instance, here is a case of each of the first three classes,

$$\partial^2_{xx}u + \partial^2_{yy}u = 0 \qquad \text{(elliptic)}$$

$$\partial^2_{xx}u - \partial^2_{yy}u = 0 \qquad \text{(hyperbolic)}$$

$$\partial^2_{xx}u - \partial_t u = 0. \qquad \text{(parabolic)}$$

When A depends on x_i, it may have multiple classes across its domain. In general, this equation and its associated initial and boundary conditions do not comprise a well-posed problem; however several special cases have been shown to be well-posed. Thus far, the most general statement of existence and uniqueness is the **cauchy-kowalevski theorem** for **cauchy problems**.

7.2 Sturm-Liouville Problems

2H

Before we introduce an important solution method for PDEs in section 7.3, we consider an *ordinary* differential equation that will arise in that method when dealing with a single spatial dimension x: the **sturm-liouville (S-L) differential equation**. Let p, q, σ be functions of x on open interval (a, b). Let X be the dependent variable and λ constant. The **regular S-L problem** is the S-L ODE[2]

$$\frac{\mathrm{d}}{\mathrm{d}x}(pX') + qX + \lambda\sigma X = 0 \tag{7.1}$$

with boundary conditions

$$\beta_1 X(a) + \beta_2 X'(a) = 0 \tag{7.2}$$

$$\beta_3 X(b) + \beta_4 X'(b) = 0 \tag{7.3}$$

with coefficients $\beta_i \in \mathbb{R}$. This is a type of **boundary value problem**.

This problem has nontrivial solutions, called **eigenfunctions** $X_n(x)$ with $n \in \mathbb{Z}_+$, corresponding to specific values of $\lambda = \lambda_n$ called **eigenvalues**.[3] There are several important theorems proven about this (see (Haberman 2018; § 5.3)). Of greatest interest to us are that

1. there exist an infinite number of eigenfunctions X_n (unique within a multiplicative constant),
2. there exists a unique corresponding *real* eigenvalue λ_n for each eigenfunction X_n,
3. the eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \cdots$,
4. eigenfunction X_n has $n - 1$ zeros on open interval (a, b),
5. the eigenfunctions X_n form an orthogonal basis with respect to weighting function σ such that any piecewise continuous function $f : [a, b] \to \mathbb{R}$ can be represented by a generalized fourier series on $[a, b]$.

This last theorem will be of particular interest in section 7.3.

2. For the S-L problem to be *regular*, it has the additional constraints that p, q, σ are continuous and $p, \sigma > 0$ on $[a, b]$. This is also sometimes called the sturm-liouville eigenvalue problem. See (Haberman 2018; § 5.3) for the more general (non-regular) S-L problem and (§ 7.4) for the multi-dimensional analog.

3. These eigenvalues are closely related to, but distinct from, the "eigenvalues" that arise in systems of linear ODEs.