7.3 PDE Solution by Separation of Variables

We are now ready to learn one of the most important techniques for solving PDEs: **separation of variables**. It applies only to **linear** PDEs since it will require the principle of superposition. Not all linear PDEs yield to this solution technique, but several that are important do.

The technique includes the following steps.

assume a product solution Assume the solution can be written as a **product solution** u_p: the product of functions of each independent variable.

separate PDE Substitute u_p into the PDE and rearrange such that at least one side of the equation has functions of a single independent variabe. If this is possible, the PDE is called **separable**.

set equal to a constant Each side of the equation depends on different independent variables; therefore, they must each equal the same constant, often called $-\lambda$.

repeat separation, as needed If there are more than two independent variables, there will be an ODE in the separated variable and a PDE (with one fewer variables) in the other independent variables. Attempt to separate the PDE until only ODEs remain.

solve each boundary value problem Solve each boundary value problem ODE, ignoring the initial conditions for now.

solve the time variable ODE Solve for the general solution of the time variable ODE, sans initial conditions.

construct the product solution Multiply the solution in each variable to construct the product solution u_p. If the boundary value problems were sturm-liouville, the product solution is a family of **eigenfunctions** from which any function can be constructed via a generalized fourier series.

apply the initial condition The product solutions individually usually do not meet the initial condition. However, a generalized fourier series of them nearly always does. **Superposition** tells us a linear combination of solutions to the PDE and boundary conditions is also a solution; the unique series that also satisfies the initial condition is the unique solution to the entire problem.

Example 7.2

Consider the one-dimensional diffusion equation PDE[a]

$$\partial_t u(t,x) = k\partial^2_{xx} u(t,x)$$

with real constant k, with dirichlet boundary conditions on inverval $x \in [0, L]$

$$u(t,0) = 0 \tag{7.8}$$

$$u(t,L) = 0, \tag{7.9}$$

and with initial condition

$$u(0,x) = f(x),$$

where f is some piecewise continuous function on $[0, L]$.

a. For more on the diffusion or heat equation, see (Haberman 2018; § 2.3), (Kreyszig 2011; § 12.5), and (Strauss 2007; § 2.3).

Assume a Product Solution First, we assume a product solution of the form $u_p(t,x) = T(t)X(x)$ where T and X are unknown functions on $t > 0$ and $x \in [0, L]$.

Separate PDE Second, we substitute the product solution into section 7.3 and separate variables:

$$T'X = kTX'' \Rightarrow$$

$$\frac{T'}{kT} = \frac{X''}{X}.$$

So it is separable! Note that we chose to group k with T, which was arbitrary but conventional.

Set Equal to a Constant Since these two sides depend on different independent variables (t and x), they must equal the same constant we call $-\lambda$, so we have two ODEs:

$$\frac{T'}{kT} = -\lambda \quad \Rightarrow T' + \lambda k T = 0$$

$$\frac{X''}{X} = -\lambda \quad \Rightarrow X'' + \lambda X = 0.$$

Solve the Boundary Value Problem The latter of these equations with the boundary conditions equation (7.8) is precisely the same sturm-liouville boundary value problem from (**ex:sturm_liouville1**), which had eigenfunctions

$$X_n(x) = \sin\left(\sqrt{\lambda_n}\, x\right) \tag{7.10}$$

$$= \sin\left(\frac{n\pi}{L} x\right) \tag{7.11}$$

with corresponding (positive) eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

Solve the Time Variable ODE The time variable ODE is homogeneous and has the familiar general solution

$$T(t) = c e^{-k\lambda t}$$

with real constant c. However, the boundary value problem restricted values of λ to λ_n, so

$$T_n(t) = ce^{-k(n\pi/L)^2 t}.$$

Construct the Product Solution The product solution is

$$\begin{aligned} u_p(t,x) &= T_n(t)X_n(x) \\ &= ce^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right). \end{aligned}$$

This is a family of solutions that each satisfy only exotically specific initial conditions.

Apply the Initial Condition The initial condition is $u(0,x) = f(x)$. The eigenfunctions of the boundary value problem form a fourier series that satisfies the initial condition on the interval $[0, L]$ if we extend f to be periodic and odd over x (Kreyszig 2011; p. 550); we call the extension f^*. The odd series synthesis can be written

$$f^*(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where the fourier analysis gives

$$b_n = \frac{2}{L}\int_0^L f^*(\chi)\sin\left(\frac{n\pi}{L}\chi\right).$$

So the complete solution is

$$u(t,x) = \sum_{n=1}^{\infty} b_n e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

Notice this satisfies the PDE, the boundary conditions, and the initial condition!

Plotting Solutions If we want to plot solutions, we need to specify an initial condition $u(0,x) = f^*(x)$ over $[0, L]$. We can choose anything piecewise continuous, but for simplicity let's let

$$f(x) = 1. \qquad (x \in [0, L])$$

The odd periodic extension is an odd square wave. The integral section 7.3 gives

$$\begin{aligned} b_n &= \frac{4}{n\pi}(1 - \cos(n\pi)) \\ &= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd.} \end{cases} \end{aligned}$$

Now we can write the solution as

$$u(t,x) = \sum_{n=1,\, n \text{ odd}}^{\infty} \frac{4}{n\pi} e^{-k(n\pi/L)^2 t} \sin\left(\frac{n\pi}{L} x\right).$$

Plotting in Python First, load some Python packages.

```
import numpy as np
import matplotlib.pyplot as plt
```

Set $k = L = 1$ and sum values for the first `N` terms of the solution.

```
L = 1
k = 1
N = 100
x = np.linspace(0,L,300)
t = np.linspace(0,2*(L/np.pi)**2,100)
u_n = np.zeros([len(t),len(x)])
for n in range(N):
  n = n+1 # because index starts at 0
  if n % 2 == 0: # even
    pass # already initialized to zeros
  else: # odd
    u_n += 4/(n*np.pi)*np.outer(
      np.exp(-k*(n*np.pi/L)**2*t),
      np.sin(n*np.pi/L*x)
    )
```

Let's first plot the initial condition.

```
fig, ax = plt.subplots()
ax.plot(x,u_n[0,:])
plt.xlabel('space $x$')
plt.ylabel('$u(0,x)$')
plt.draw()
```

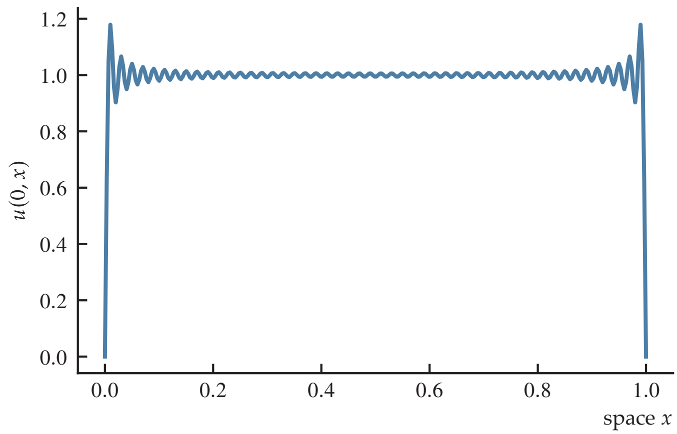

Figure 7.2. Initial condition $u(0,x)$

Now we plot the entire response.

```
fig, ax = plt.subplots()
plt.contourf(t,x,u_n.T)
c = plt.colorbar()
c.set_label('$u(t,x)$')
plt.xlabel('time $t$')
plt.ylabel('space $x$')
plt.show()
```

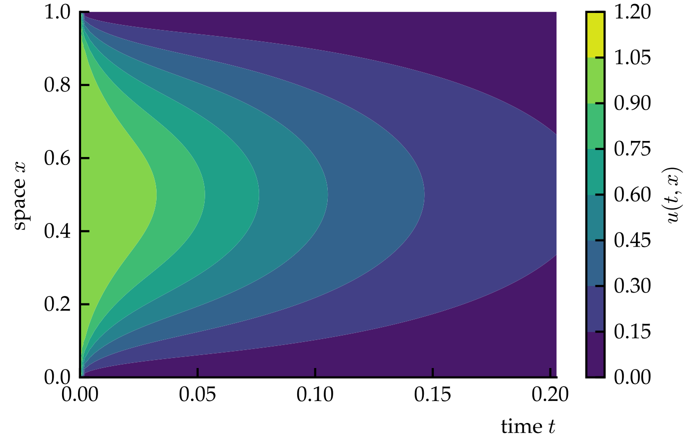

Figure 7.3. Solution $u(t, x)$

We see the diffusive action proceeds as we expected.

7.4 The 1D Wave Equation

2I

The one-dimensional **wave equation** is the linear PDE

$$\partial_{tt}^2 u(t, x) = c^2 \partial_{xx}^2 u(t, x).$$

with real constant c. This equation models such phenomena as strings, fluids, sound, and light. It is subject to initial and boundary conditions and can be extended to multiple spatial dimensions. For 2D and 3D examples in rectangular and polar coordinates, see (Kreyszig 2011; § 12.9 12.10) and (Haberman 2018; § 4.5 7.3).

Example 7.3

Consider the one-dimensional wave equation PDE

$$\partial_{tt}^2 u(t, x) = c^2 \partial_{xx}^2 u(t, x) \tag{7.12}$$

with real constant c and with dirichlet boundary conditions on inverval $x \in [0, L]$

$$u(t, 0) = 0 \quad \text{and} \quad u(t, L) = 0, \tag{7.13}$$

and with initial conditions (we need two because of the second time-derivative)

$$u(0, x) = f(x) \quad \text{and} \quad \partial_t u(0, x) = g(x),$$

where f and g are some piecewise continuous functions on $[0, L]$.