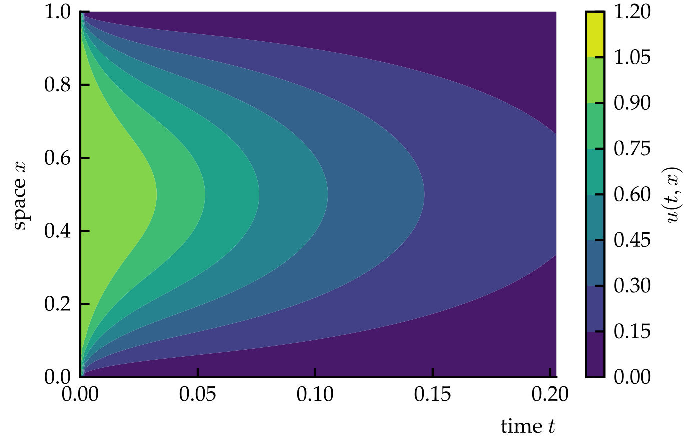


Figure 7.3. Solution $u(t, x)$

We see the diffusive action proceeds as we expected.

## 7.4 The 1D Wave Equation

The one-dimensional **wave equation** is the linear PDE

$$\partial_{tt}^2 u(t, x) = c^2 \partial_{xx}^2 u(t, x).$$

with real constant $c$. This equation models such phenomena as strings, fluids, sound, and light. It is subject to initial and boundary conditions and can be extended to multiple spatial dimensions. For 2D and 3D examples in rectangular and polar coordinates, see (Kreyszig 2011; § 12.9 12.10) and (Haberman 2018; § 4.5 7.3).

**Example 7.3**

Consider the one-dimensional wave equation PDE

$$\partial_{tt}^2 u(t, x) = c^2 \partial_{xx}^2 u(t, x) \tag{7.12}$$

with real constant $c$ and with dirichlet boundary conditions on inverval $x \in [0, L]$

$$u(t, 0) = 0 \quad \text{and} \quad u(t, L) = 0, \tag{7.13}$$

and with initial conditions (we need two because of the second time-derivative)

$$u(0, x) = f(x) \quad \text{and} \quad \partial_t u(0, x) = g(x),$$

where $f$ and $g$ are some piecewise continuous functions on $[0, L]$.

**Assume a Product Solution** First, we assume a product solution of the form $u_p(t,x) = T(t)X(x)$ where $T$ and $X$ are unknown functions on $t > 0$ and $x \in [0, L]$.

**Separate PDE** Second, we substitute the product solution into equation (7.12) and separate variables:

$$T''X = c^2 T X'' \Rightarrow$$
$$\frac{T''}{c^2 T} = \frac{X''}{X}.$$

So it is separable! Note that we chose to group $c$ with $T$, which was arbitrary but conventional.

**Set Equal to a Constant** Since these two sides depend on different independent variables ($t$ and $x$), they must equal the same constant we call $-\lambda$, so we have two ODEs:

$$\frac{T''}{c^2 T} = -\lambda \quad \Rightarrow T'' + \lambda c^2 T = 0$$
$$\frac{X''}{X} = -\lambda \quad \Rightarrow X'' + \lambda X = 0.$$

**Solve the Boundary Value Problem** The latter of these equations with the boundary conditions **??** is precisely the same sturm-liouville boundary value problem from **??**, which had eigenfunctions

$$X_n(x) = \sin\left(\sqrt{\lambda_n} x\right) \tag{7.14}$$
$$= \sin\left(\frac{n\pi}{L} x\right) \tag{7.15}$$

with corresponding (positive) eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

**Solve the Time Variable ODE** The time variable ODE is homogeneous and, with $\lambda$ restricted by the reals by the boundary value problem, has the familiar general solution

$$T(t) = k_1 \cos(c\sqrt{\lambda} t) + k_2 \sin(c\sqrt{\lambda} t)$$

with real constants $k_1$ and $k_2$. However, the boundary value problem restricted values of $\lambda$ to $\lambda_n$, so

$$T_n(t) = k_1 \cos\left(\frac{cn\pi}{L} t\right) + k_2 \sin\left(\frac{cn\pi}{L} t\right).$$

**Construct the Product Solution** The product solution is

$$\begin{aligned} u_p(t,x) &= T_n(t)X_n(x) \\ &= k_1 \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{cn\pi}{L}t\right) + k_2 \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{cn\pi}{L}t\right). \end{aligned}$$

This is a family of solutions that each satisfy only exotically specific initial conditions.

**Apply the Initial Conditions** Recall that superposition tells us that any linear combination of the product solution is also a solution. Therefore,

$$u(t,x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{cn\pi}{L}t\right)$$

is a solution. If $a_n$ and $b_n$ are properly selected to satisfy the initial conditions, section 7.4 will be the solution to the entire problem. Substituting $t = 0$ into our potential solution gives

$$u(0,x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) \tag{7.16}$$

$$\partial_t u(t,x)|_{t=0} = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right). \tag{7.17}$$

Let us extend $f$ and $g$ to be periodic and odd over $x$; we call the extensions $f^*$ and $g^*$. From equation (7.16), the intial conditions are satsified if

$$f^*(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) \tag{7.18}$$

$$g^*(x) = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right). \tag{7.19}$$

We identify these as two odd fourier syntheses. The corresponding fourier analyses are

$$a_n = \frac{2}{L}\int_0^L f^*(\chi)\sin\left(\frac{n\pi}{L}\chi\right) \tag{7.20}$$

$$b_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g^*(\chi)\sin\left(\frac{n\pi}{L}\chi\right) \tag{7.21}$$

So the complete solution is equations (7.18) and (7.19) with components given by equations (7.20) and (7.21). Notice this satisfies the PDE, the boundary conditions, and the initial condition!

**Discussion** It can be shown that this series solution is equivalent to two *traveling waves* that are interfering (see (Haberman 2018; § 4.4) and (Kreyszig 2011; § 12.2)). This is convenient because computing the series solution exactly requires an infinite summation. We show in the following section that the approximation by partial summation is still quite good.

**Choosing Specific Initial Conditions** If we want to plot solutions, we need to specify initial conditions over $[0, L]$. Let's model a string being suddenly struck from rest as

$$f(x) = 0$$
$$g(x) = \delta(x - \Delta L)$$

where $\delta$ is the dirac delta distribution and $\Delta \in [0, L]$ is a fraction of $L$ representing the location of the string being struck. The odd periodic extension is an odd pulse train. The integrals of equations (7.20) and (7.21) give

$$a_n = 0 \tag{7.22}$$
$$b_n = \frac{2}{cn\pi} \int_0^L \delta(\chi - \Delta L) \sin\left(\frac{n\pi}{L}\chi\right) \mathrm{d}x$$
$$= \frac{2}{cn\pi} \sin(n\pi\Delta). \qquad \text{(sifting property)}$$

Now we can write the solution as

$$u(t, x) = \sum_{n=1}^{\infty} \frac{2}{cn\pi} \sin(n\pi\Delta) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{cn\pi}{L}t\right).$$

**Plotting in Python** First, load some Python packages.

```
import numpy as np
import matplotlib.pyplot as plt
```

Set $c = L = 1$ and sum values for the first `N` terms of the solution for some striking location $\Delta$.

```
Delta = 0.1 # 0 <= Delta <= L
L = 1
c = 1
N = 150
t = np.linspace(0,30*(L/np.pi)**2,100)
x = np.linspace(0,L,150)
t_b, x_b = np.meshgrid(t,x)
u_n = np.zeros([len(x),len(t)])
for n in range(N):
  n = n+1 # because index starts at 0
  u_n += 4/(c*n*np.pi)* \
    np.sin(n*np.pi*Delta)* \
    np.sin(c*n*np.pi/L*t_b)* \
    np.sin(n*np.pi/L*x_b)
```

Let's first plot some early snapshots of the response.

```
import seaborn as sns
n_snaps = 7
sns.set_palette(
  sns.diverging_palette(
    240, 10, n=n_snaps, center="dark"
  )
)
fig, ax = plt.subplots()
it = np.linspace(2,77,n_snaps,dtype=int)
for i in range(len(it)):
  ax.plot(x,u_n[:,it[i]],label=f"t = {t[i]:.3f}");
lgd = ax.legend(
  bbox_to_anchor=(1.05, 1),
  loc='upper left'
)
plt.xlabel('space $x$')
plt.ylabel('$u(t,x)$')
plt.draw()
```

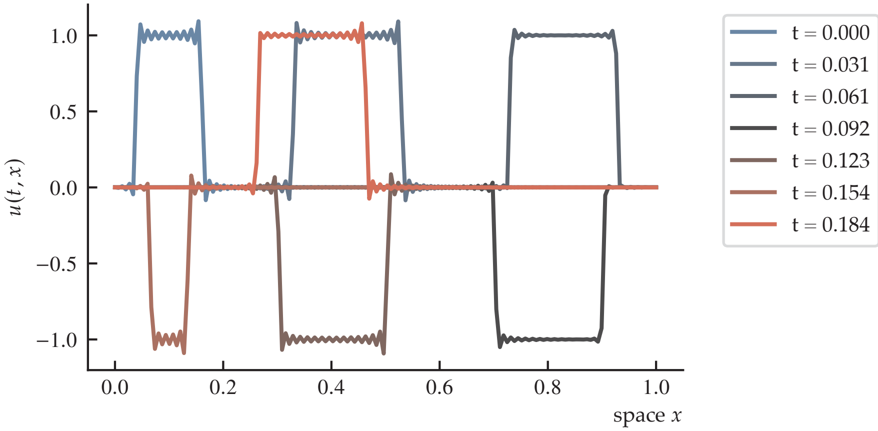


Figure 7.4. Early snapshots of $u(t, x)$.

Now we plot the entire response.

```
fig, ax = plt.subplots()
p = ax.contourf(t,x,u_n)
c = fig.colorbar(p,ax=ax)
c.set_label('$u(t,x)$')
plt.xlabel('time $t$')
plt.ylabel('space $x$')
plt.show()
```

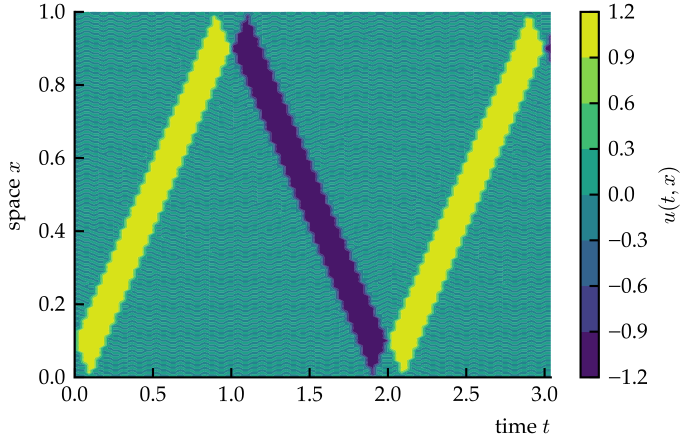


Figure 7.5. Solution $u(t, x)$.

We see a wave develop and travel, reflecting and inverting off each boundary.

## 7.5 Problems

B2

**Problem 7.1** HORTICULTURE The PDE of example 7.2 can be used to describe the conduction of heat along a long, thin rod, insulated along its length, where $u(t,x)$ represents temperature. The initial and dirichlet boundary conditions in that example would be interpreted as an initial temperature distribution along the bar and fixed temperatures of the ends. Now consider the same PDE

$$\partial_t u(t,x) = k\partial^2_{xx} u(t,x) \tag{7.23}$$

with real constant $k$, with mixed boundary conditions on interval $x \in [0, L]$

$$u(t,0) = 0 \tag{7.24a}$$

$$\partial_x u(t,x)|_{x=L} = 0, \tag{7.24b}$$

and with initial condition

$$u(0,x) = f(x), \tag{7.25}$$

where $f$ is some piecewise continuous function on $[0, L]$. This represents the insulation of one end ($L$) of the rod and the other end (0) is held at a fixed temperature.

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions $X_n$ and eigenvalues $\lambda_n$.
c. Solve for the general solution of the time variable ODE.
d. Write the product solution and apply the initial condition $f(x)$ by constructing it from a generalized fourier series of the product solution.
e. Let $L = k = 1$ and

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, L/2) \\ 100 & \text{for } x \in [L/2, L] \end{cases} \tag{7.26}$$

as shown in figure 7.6. Compute the solution series components. Plot the sum of the first 50 terms over $x$ and $t$.