7.5 Problems

B2

Problem 7.1 HORTICULTURE The PDE of example 7.2 can be used to describe the conduction of heat along a long, thin rod, insulated along its length, where $u(t,x)$ represents temperature. The initial and dirichlet boundary conditions in that example would be interpreted as an initial temperature distribution along the bar and fixed temperatures of the ends. Now consider the same PDE

$$\partial_t u(t,x) = k\partial^2_{xx} u(t,x) \tag{7.23}$$

with real constant k, with mixed boundary conditions on interval $x \in [0, L]$

$$u(t,0) = 0 \tag{7.24a}$$

$$\partial_x u(t,x)|_{x=L} = 0, \tag{7.24b}$$

and with initial condition

$$u(0,x) = f(x), \tag{7.25}$$

where f is some piecewise continuous function on $[0, L]$. This represents the insulation of one end (L) of the rod and the other end (0) is held at a fixed temperature.

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions X_n and eigenvalues λ_n.
c. Solve for the general solution of the time variable ODE.
d. Write the product solution and apply the initial condition $f(x)$ by constructing it from a generalized fourier series of the product solution.
e. Let $L = k = 1$ and

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, L/2) \\ 100 & \text{for } x \in [L/2, L] \end{cases} \tag{7.26}$$

as shown in figure 7.6. Compute the solution series components. Plot the sum of the first 50 terms over x and t.

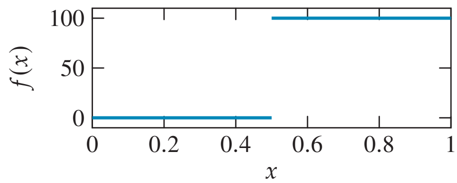

Figure 7.6. Initial condition for problem 7.1.

Problem 7.2 POLTERGEIST The PDE of example 7.2 can be used to describe the conduction of heat along a long, thin rod, insulated along its length, where $u(t, x)$ represents temperature. The initial and dirichlet boundary conditions in that example would be interpreted as an initial temperature distribution along the bar and fixed temperatures of the ends. Now consider the same PDE

$$\partial_t u(t, x) = k\partial^2_{xx} u(t, x) \tag{7.27}$$

with real constant k, now with *neumann* boundary conditions on interval $x \in [0, L]$

$$\partial_x u|_{x=0} = 0 \quad \text{and} \quad \partial_x u|_{x=L} = 0, \tag{7.28a}$$

and with initial condition

$$u(0, x) = f(x), \tag{7.29}$$

where f is some piecewise continuous function on $[0, L]$. This represents the complete insulation of the ends of the rod, such that no heat flows from the ends (or from anywhere else).

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions X_n and eigenvalues λ_n.
c. Solve for the general solution of the time variable ODE.
d. Write the product solution and apply the initial condition $f(x)$ by constructing it from a generalized fourier series of the product solution.
e. Let $L = k = 1$ and $f(x) = 100 - 200/L\,|x - L/2|$ as shown in figure 7.7. Compute the solution series components. Plot the sum of the first 50 terms over x and t.

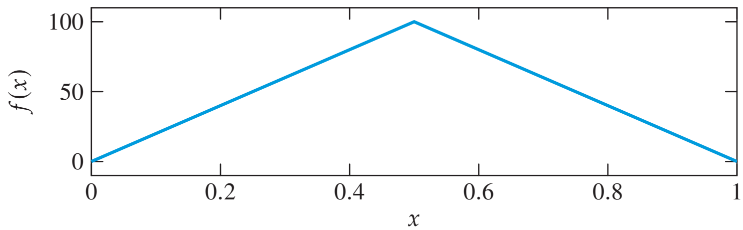

Figure 7.7. Initial condition for problem 7.2.

Problem 7.3 KATHMANDU Consider the free vibration of a uniform and relatively thin beam—with modulus of elasticity E, second moment of cross-sectional area I, and mass-per-length μ—pinned at each end. The PDE describing this is a version of the euler-bernoulli beam equation for vertical motion u:

$$\partial^2_{tt} u(t, x) = -\alpha^2 \partial^4_{xxxx} u(t, x) \tag{7.30}$$

with real constant α defined as

$$\alpha^2 = \frac{EI}{\mu}. \tag{7.31}$$

Pinned supports fix vertical motion such that we have boundary conditions on interval $x \in [0, L]$

$$u(t, 0) = 0 \quad \text{and} \quad u(t, L) = 0. \tag{7.32a}$$

Additionally, pinned supports cannot provide a moment, so

$$\partial^2_{xx} u|_{x=0} = 0 \quad \text{and} \quad \partial^2_{xx} u|_{x=L} = 0. \tag{7.32b}$$

Furthermore, consider the initial conditions

$$u(0, x) = f(x), \quad \text{and} \quad \partial_t u|_{t=0} = 0. \tag{7.33a}$$

where f is some piecewise continuous function on $[0, L]$.

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions X_n and eigenvalues λ_n. Assume real $\lambda > 0$ (it's true but tedious to show).
c. Solve for the general solution of the time variable ODE.
d. Write the product solution and apply the initial conditions by constructing it from a generalized fourier series of the product solution.
e. Let $L = \alpha = 1$ and $f(x) = \sin(10\pi x/L)$ as shown in figure 7.8. Compute the solution series components. Plot the sum of the first 50 terms over x and t.

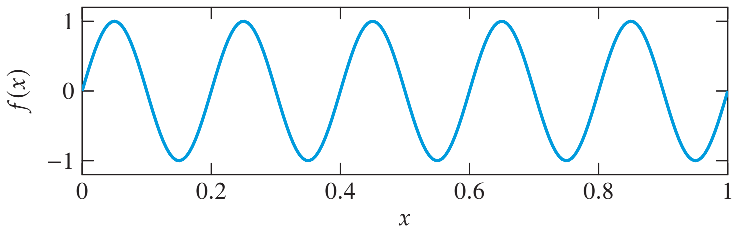

Figure 7.8. Initial condition for problem 7.3.

Problem 7.4 HURRIED Given the 1D heat equation,

$$\frac{\partial}{\partial t}u(t,x) = \alpha\frac{\partial^2}{\partial x^2}u(t,x),$$

with boundary conditions,

$$\frac{\partial}{\partial x}u(t,x)\Big|_{x=L} = 0$$
$$u(t,0) = 0,$$

and initial condition,

$$u(0,x) = \begin{cases} 1 & \frac{L}{3} \leq x \leq \frac{2L}{3} \\ 0 & \text{otherwise} \end{cases}$$

a. show that this PDE is separable,
b. solve the sturm-liouville boundary condition problem,
c. find the fourier coefficients, and
d. given $L = 1$, $\alpha = 1$, and using the first 100 terms of the infinite sum, plot the solution at $t = 0$, $t = 0.01$, and $t = 0.1$.

Problem 7.5 PLUCK Consider the one-dimensional wave equation PDE

$$\partial^2_{tt}u(t,x) = c^2\partial^2_{xx}u(t,x) \tag{7.34}$$

with real constant c and with dirichlet boundary conditions on inverval $x \in [0,L]$

$$u(t,0) = 0 \quad \text{and} \quad u(t,L) = 0, \tag{7.35a}$$

and with initial conditions (we need two because of the second time-derivative)

$$u(0,x) = f(x) \quad \text{and} \quad \partial_t u(0,x) = g(x), \tag{7.36}$$

where f and g are some piecewise continuous functions on $[0,L]$.

Assume we can model a musical instrument's plucked string with equations (7.34) to (7.36) with the initial velocity $g(x) = 0$ and initial displacement $f(x)$ given in figure 7.9.

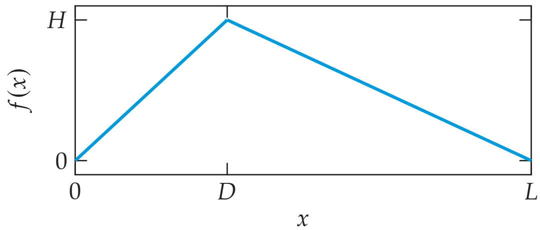

Figure 7.9. Initial condition for problem 7.5.

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions X_n and eigenvalues λ_n.
c. Solve for the general solution of the time variable ODE.
d. Write the product solution and apply the initial conditions by constructing it from a generalized fourier series of the product solution.
e. Let $H = 0.5$, $L = c = 1$, and $D = 0.3$. Compute the solution series components. Plot the sum of the first 50 terms over x and t.

8 Optimization

EW

This chapter concerns optimization mathematics.

8.1 Gradient Descent

Y7

Consider a multivariate function $f : \mathbb{R}^n \to \mathbb{R}$ that represents some cost or value. This is called an **objective function**, and we often want to find an $\boldsymbol{X} \in \mathbb{R}^n$ that yields f's **extremum**: minimum or maximum, depending on whichever is desirable.

It is important to note however that some functions have no finite extremum. Other functions have multiple. Finding a **global extremum** is generally difficult; however, many good methods exist for finding a **local extremum**: an extremum for some region $R \subset \mathbb{R}^n$.

The method explored here is called **gradient descent**. It will soon become apparent why it has this name.

8.1.1 Stationary Points

Recall from basic calculus that a function f of a single variable had potential local extrema where $\mathrm{d}f(x)/\mathrm{d}x = 0$. The multivariate version of this, for multivariate function f, is

$$\operatorname{grad} f = \mathbf{0}.$$

A value $\boldsymbol{X}$ for which section 8.1.1 holds is called a **stationary point**. However, as in the univariate case, a stationary point may not be a local extremum; in these cases, it called a **saddle point**.

Consider the **hessian matrix** H with values, for independent variables x_i,

$$H_{ij} = \partial^2_{x_i x_j} f.$$

For a stationary point $\boldsymbol{X}$, the **second partial derivative test** tells us if it is a local maximum, local minimum, or saddle point: