8.4 Problems

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Problem 8.1 CHORTLE Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined as

$$f(\boldsymbol{x}) = \cos(x_1 - e^{x_2} + 2)\sin(x_1^2/4 - x_2^2/3 + 4) \tag{8.14}$$

Use the method of Barzilai and Borwein (1988) starting at $\boldsymbol{x}_0 = (1, 1)$ to find a minimum of the function.

Problem 8.2 CUMMERBUND Consider the functions (a) $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and (b) $f_2 : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f_1(\boldsymbol{x}) = 4(x_1 - 16)^2 + (x_2 + 64)^2 + x_1 \sin^2 x_1 \tag{8.15}$$

$$f_2(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x} \cdot A\boldsymbol{x} - \boldsymbol{b} \cdot \boldsymbol{x} \tag{8.16}$$

where

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 15 \end{bmatrix} \quad \text{and} \tag{8.17a}$$

$$b = \begin{bmatrix} -2 & 1 \end{bmatrix}^\top. \tag{8.17b}$$

Use the method of Barzilai and Borwein (1988) starting at some $\boldsymbol{x}_0$ to find a minimum of each function.

Problem 8.3 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(\boldsymbol{x}) = \sin x_1 + \cos x_2 + \sqrt{(x_1 - 2)^2 + (x_2 + 1)^2}. \tag{8.18}$$

Use the gradient descent method of Barzilai and Borwein (1988) with a step size of $T = 10^{-8}$ starting at (a) $\boldsymbol{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ and (b) $\boldsymbol{x}_0' = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top$ to find minima of f. (c) Explain why the two minima are different.

Problem 8.4 MELTY Maximize the objective function

$$f(\boldsymbol{x}) = \boldsymbol{c} \cdot \boldsymbol{x} \tag{8.19a}$$

for $\boldsymbol{x} \in \mathbb{R}^3$ and

$$\boldsymbol{c} = \begin{bmatrix} 3 & -8 & 1 \end{bmatrix}^\top \tag{8.19b}$$

subject to constraints

$$0 \le x_1 \le 20 \tag{8.20a}$$

$$-5 \le x_2 \le 0 \tag{8.20b}$$

$$5 \le x_3 \le 17 \tag{8.20c}$$

$$x_1 + 4x_2 \le 50 \tag{8.20d}$$

$$2x_1 + x_3 \le 43 \tag{8.20e}$$

$$-4x_1 + x_2 - 5x_3 \ge -99. \tag{8.20f}$$

Problem 8.5 LATENESS Using gradient decent find the minimum of the function,

$$f(x) = x_1^2 + x_2^2 - \frac{x_1}{10} + \cos(2x_1),$$

starting at the location $x = [-0.5,\ 0.75]^T$, and with a constant value $\alpha = 0.01$.

a. What is the location of the minimum you found?
b. Is this location the global minimum?

9 Nonlinear Analysis

The ubiquity of near-linear systems and the tools we have for analyses thereof can sometimes give the impression that nonlinear systems are exotic or even downright flamboyant. However, a great many systems[1] important for a mechanical engineer are frequently hopelessly nonlinear. Here are a some examples of such systems.

- A robot arm.
- Viscous fluid flow (usually modelled by the navier-stokes equations).
- Nonequilibrium thermodynamics.
- Anything that “fills up” or “saturates.”
- Nonlinear optics.
- Einstein’s field equations (gravitation in general relativity).
- Heat radiation and nonlinear heat conduction.
- Fracture mechanics.
- The 3-body problem.

Lest we think this is merely an inconvenience, we should keep in mind that it is actually the nonlinearity that makes many phenomena useful. For instance, the laser depends on the nonlinearity of its optics. Similarly, transistors and the digital circuits made thereby (including the microprocessor) wouldn’t function if their physics were linear.

In this chapter, we will see some ways to formulate, characterize, and simulate nonlinear systems. Purely analytic techniques are few for nonlinear systems. Most are beyond the scope of this text, but we describe a few, mostly in (**lec:nonlinear-system-characteristics**). Simulation via numerical integration of nonlinear dynamical equations is the most accessible technique, so it is introduced.

We skip a discussion of linearization; of course, if this is an option, it is preferable. Instead, we focus on the nonlinearizable.

1. As is customary, we frequently say “system” when we mean “mathematical system model.” Recall that multiple models may be used for any given physical system, depending on what one wants to know.

For a good introduction to nonlinear dynamics, see (Strogatz and Dichter 2016). A more engineer-oriented introduction is (Kolk and Lerman 1993).

9.1 Nonlinear State-Space Models

A state-space model has the general form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{9.1}$$

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{9.2}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are vector-valued functions that depend on the system. **Nonlinear state-space models** are those for which $\boldsymbol{f}$ is a nonlinear functional of either $\boldsymbol{x}$ or $\boldsymbol{u}$. For instance, a state variable x_1 might appear as x_1^2 or two state variables might combine as $x_1 x_2$ or an input u_1 might enter the equations as $\log u_1$.

9.1.1 Autonomous and Nonautonomous Systems

An **autonomous system** is one for which $\boldsymbol{f}(\boldsymbol{x})$, with neither time nor input appearing explicitly. A **nonautonomous system** is one for which either t or $\boldsymbol{u}$ *do* appear explicitly in $\boldsymbol{f}$. It turns out that we can always write nonautonomous systems as autonomous by substituting in $\boldsymbol{u}(t)$ and introducing an extra state variable for t (Strogatz and Dichter 2016).

Therefore, without loss of generality, we will focus on ways of analyzing autonomous systems.

9.1.2 Equilibrium

An **equilibrium state** (also called a stationary point) $\overline{\boldsymbol{x}}$ is one for which $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{0}$. In most cases, this occurs only when the input $\boldsymbol{u}$ is a constant $\overline{\boldsymbol{u}}$ and, for time-varying systems, at a given time $\overline{t}$. For autonomous systems, equilibrium occurs when the following holds:

$$\boldsymbol{f}(\overline{\boldsymbol{x}}) = \boldsymbol{0}.$$

This is a system of nonlinear algebraic equations, which can be challenging to solve for $\overline{\boldsymbol{x}}$. However, frequently, several solutions—that is, equilibrium states—do exist.

9.2 Nonlinear System Characteristics

Characterizing nonlinear systems can be challenging without the tools developed for linear system characterization. However, there are ways of characterizing nonlinear systems, and we'll here explore a few.