For a good introduction to nonlinear dynamics, see (Strogatz and Dichter 2016). A more engineer-oriented introduction is (Kolk and Lerman 1993).

9.1 Nonlinear State-Space Models

A state-space model has the general form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{9.1}$$

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{9.2}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are vector-valued functions that depend on the system. **Nonlinear state-space models** are those for which $\boldsymbol{f}$ is a nonlinear functional of either $\boldsymbol{x}$ or $\boldsymbol{u}$. For instance, a state variable x_1 might appear as x_1^2 or two state variables might combine as $x_1 x_2$ or an input u_1 might enter the equations as $\log u_1$.

9.1.1 Autonomous and Nonautonomous Systems

An **autonomous system** is one for which $\boldsymbol{f}(\boldsymbol{x})$, with neither time nor input appearing explicitly. A **nonautonomous system** is one for which either t or $\boldsymbol{u}$ *do* appear explicitly in $\boldsymbol{f}$. It turns out that we can always write nonautonomous systems as autonomous by substituting in $\boldsymbol{u}(t)$ and introducing an extra state variable for t (Strogatz and Dichter 2016).

Therefore, without loss of generality, we will focus on ways of analyzing autonomous systems.

9.1.2 Equilibrium

An **equilibrium state** (also called a stationary point) $\overline{\boldsymbol{x}}$ is one for which $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{0}$. In most cases, this occurs only when the input $\boldsymbol{u}$ is a constant $\overline{\boldsymbol{u}}$ and, for time-varying systems, at a given time $\overline{t}$. For autonomous systems, equilibrium occurs when the following holds:

$$\boldsymbol{f}(\overline{\boldsymbol{x}}) = \boldsymbol{0}.$$

This is a system of nonlinear algebraic equations, which can be challenging to solve for $\overline{\boldsymbol{x}}$. However, frequently, several solutions—that is, equilibrium states—do exist.

9.2 Nonlinear System Characteristics

Characterizing nonlinear systems can be challenging without the tools developed for linear system characterization. However, there are ways of characterizing nonlinear systems, and we'll here explore a few.