

## sets.setintro Introduction to set theory

**Set theory** is the language of the modern foundation of mathematics, as discussed in [Lec. itself.found](#). It is unsurprising, then, that it arises throughout the study of mathematics. We will use set theory extensively in ?? on probability theory.

The axioms of ZFC set theory were introduced in [Lec. itself.found](#). Instead of proceeding in the pure mathematics way of introducing and proving theorems, we will opt for a more applied approach in which we begin with some simple definitions and include basic operations. A more thorough and still readable treatment is given by Ciesielski (1997) and a very gentle version by Enderton (1977).

A **set** is a collection of objects. Set theory gives us a way to describe these collections. Often, the objects in a set are numbers or sets of numbers. However, a set could represent collections of zebras and trees and hairballs. For instance, here are some sets:



A **field** is a set with special structure. This structure is provided by the **addition** (+) and **multiplication** ( $\times$ ) operators and their inverses **subtraction** ( $-$ ) and **division** ( $\div$ ). The quintessential example of a field is the set of **real numbers**  $\mathbb{R}$ , which admits these operators, making it a field. The reals  $\mathbb{R}$ , the complex numbers  $\mathbb{C}$ , the integers  $\mathbb{Z}$ , and the natural numbers<sup>1</sup>  $\mathbb{N}$  are the fields we typically consider.

**Set membership** is the belonging of an object to a set. It is denoted with the symbol  $\in$ , which can be read "is an element of," for element  $x$  and set  $X$ :

1. When the natural numbers include zero, we write  $\mathbb{N}_0$ .

For instance, we might say  $7 \in \{1, 7, 2\}$  or  $4 \notin \{1, 7, 2\}$ . Or, we might declare that  $a$  is a real number by stating:  $x \in \mathbb{R}$ .

**Set operations** can be used to construct new sets from established sets. We consider a few common set operations, now.

The **union**  $\cup$  of sets is the set containing all the elements of the original sets (no repetition allowed). The union of sets  $A$  and  $B$  is denoted  $A \cup B$ . For instance, let  $A = \{1, 2, 3\}$  and  $B = \{-1, 3\}$ ; then

The **intersection**  $\cap$  of sets is a set containing the elements common to all the original sets. The intersection of sets  $A$  and  $B$  is denoted  $A \cap B$ . For instance, let  $A = \{1, 2, 3\}$  and  $B = \{2, 3, 4\}$ ; then

If two sets have no elements in common, the intersection is the **empty set**  $\emptyset = \{\}$ , the unique set with no elements.

The **set difference** of two sets  $A$  and  $B$  is the set of elements in  $A$  that aren't also in  $B$ . It is denoted  $A \setminus B$ . For instance, let  $A = \{1, 2, 3\}$  and  $B = \{2, 3, 4\}$ . Then

A **subset**  $\subseteq$  of a set is a set, the elements of which are contained in the original set. If the two sets are equal, one is still considered a subset of the other. We call a subset that is not

equal to the other set a **proper subset**  $\subset$ . For instance, let  $A = \{1, 2, 3\}$  and  $B = \{1, 2\}$ . Then

The **complement** of a subset is a set of elements of the original set that aren't in the subset. For instance, if  $B \subseteq A$ , then the complement of  $B$ , denoted  $\bar{B}$  is

The **cartesian product** of two sets  $A$  and  $B$  is denoted  $A \times B$  and is the set of all ordered pairs  $(a, b)$  where  $a \in A$  and  $b \in B$ . It's worthwhile considering the following notation for this definition:

which means "the cartesian product of  $A$  and  $B$  is the ordered pair  $(a, b)$  such that  $a \in A$  and  $b \in B$ " in **set-builder notation** (Wikipedia, 2019k).

Let  $A$  and  $B$  be sets. A **map** or **function**  $f$  from  $A$  to  $B$  is an assignment of some element  $a \in A$  to each element  $b \in B$ . The function is denoted  $f : A \rightarrow B$  and we say that  $f$  maps each element  $a \in A$  to an element  $f(a) \in B$  called the **value** of  $a$  under  $f$ , or  $a \mapsto f(a)$ . We say that  $f$  has **domain**  $A$  and **codomain**  $B$ . The **image** of  $f$  is the subset of its codomain  $B$  that contains the values of all elements mapped by  $f$  from its domain  $A$ .