vecs.curl Curl, line integrals, and circulation

Line integrals

Consider a curve C in a Euclidean vector space $\mathbb{R}^3$. Let $\mathbf{r}(t) = [x(t), y(t), z(t)]$ be a parametric representation of C. Furthermore, let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector-valued function of $\mathbf{r}$ and let $\mathbf{r}'(t)$ be the tangent vector. The **line integral** is

$$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \tag{1}$$

which integrates $\mathbf{F}$ along the curve. For more on computing line integrals, see Schey (2005, pp. 63-74) and Kreyszig (2011, § 10.1, 10.2).
If $\mathbf{F}$ is a **force** being applied to an object moving along the curve C, the line integral is the **work** done by the force. More generally, the line integral integrates $\mathbf{F}$ along the tangent of C.

Circulation

Consider the line integral over a closed curve C, denoted by

$$\oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{2}$$

We call this quantity the **circulation** of $\mathbf{F}$ around C.
For certain vector-valued functions $\mathbf{F}$, the circulation is zero for every curve. In these cases (static electric fields, for instance), this is sometimes called the **the law of circulation**.

Curl

Consider the division of the circulation around a curve in $\mathbb{R}^3$ by the surface area it encloses ΔS,

$$\frac{1}{\Delta S} \oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{3}$$

In a manner analogous to the operation that gaves us the divergence, let's consider shrinking this curve to a point and the surface area to zero,

$$\lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{4}$$

We call this quantity the "scalar" **curl** of $\mathbf{F}$ at each point in $\mathbb{R}^3$ *in the direction normal to* ΔS as it shrinks to zero. Taking three (or n for $\mathbb{R}^n$) "scalar" curls in indepedent normal directions (enough to span the vector space), we obtain the **curl** proper, which is a vector-valued function $\text{curl} : \mathbb{R}^3 \to \mathbb{R}^3$.

The curl is coordinate-independent. In cartesian coordinates, it can be shown to be equivalent to the following.

Equation 5 curl: differential form, cartesian coordinates

$$\text{curl}\, \mathbf{F} = \begin{bmatrix} \partial_y F_z - \partial_z F_y & \partial_z F_x - \partial_x F_z & \partial_x F_y - \partial_y F_x \end{bmatrix}^\top$$

But what does the curl of $\mathbf{F}$ represent? It quantifies the local rotation of $\mathbf{F}$ about each point. If $\mathbf{F}$ represents a fluid's velocity, $\text{curl}\, \mathbf{F}$ is the local rotation of the fluid about each point and it is called the **vorticity**.

Zero curl, circulation, and path independence

Circulation

It can be shown that if the circulation of $\mathbf{F}$ on all curves is zero, then in each direction $\mathbf{n}$ and at every point $\text{curl}\, \mathbf{F} = 0$ (i.e. $\mathbf{n} \cdot \text{curl}\, \mathbf{F} = 0$).

Conversely, for $\text{curl}\, \mathbf{F} = 0$ *in a simply connected region*[2], $\mathbf{F}$ has *zero circulation*.

Succinctly, informally, and without the requisite

2. A region is simply connected if every curve in it can shrink to a point without leaving the region. An example of a region that is not simply connected is the surface of a toroid.

qualifiers above,

$$\text{zero circulation} \Rightarrow \text{zero curl} \tag{6}$$

$$\text{zero curl } + \text{simply connected region} \Rightarrow \text{zero circulation.} \tag{7}$$

Path independence

It can be shown that if the path integral of $\mathbf{F}$ on all curves between any two points is **path-independent**, then in each direction $\mathbf{n}$ and at every point $\operatorname{curl}\mathbf{F} = 0$ (i.e. $\mathbf{n} \cdot \operatorname{curl}\mathbf{F} = 0$). Conversely, for $\operatorname{curl}\mathbf{F} = 0$ in a simply connected region, all line integrals are independent of path. Succinctly, informally, and without the requisite qualifiers above,

$$\text{path independence} \Rightarrow \text{zero curl} \tag{8}$$

$$\text{zero curl } + \text{simply connected region} \Rightarrow \text{path independence.} \tag{9}$$

… and how they relate

It is also true that

$$\text{path independence} \Leftrightarrow \text{zero circulation.} \tag{10}$$

So, putting it all together, we get Fig. curl.1.

Exploring curl

Curl is perhaps best explored by considering it for a vector field in $\mathbb{R}^2$. Such a field in cartesian coordinates $\mathbf{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ has curl

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{bmatrix}\partial_y 0 - \partial_z F_y & \partial_z F_x - \partial_x 0 & \partial_x F_y - \partial_y F_x\end{bmatrix}^\top \\ &= \begin{bmatrix}0 - 0 & 0 - 0 & \partial_x F_y - \partial_y F_x\end{bmatrix}^\top \\ &= \begin{bmatrix}0 & 0 & \partial_x F_y - \partial_y F_x\end{bmatrix}^\top .\end{aligned} \tag{11}$$

That is, $\operatorname{curl}\mathbf{F} = (\partial_x F_y - \partial_y F_x)\hat{k}$ and the only nonzero component is normal to the xy-plane. If

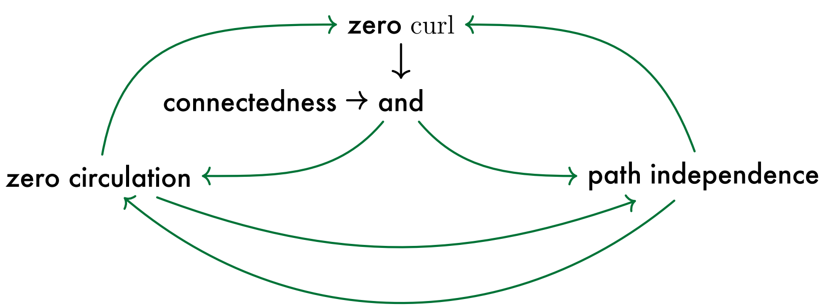

Figure curl.1: an *implication graph* relating zero curl, zero circulation, path independence, and connectedness. Green edges represent implication (a implies b) and black edges represent logical conjunctions.

we overlay a quiver plot of $\mathbf{F}$ over a "color density" plot representing the $\hat{k}$-component of $\operatorname{curl} \mathbf{F}$, we can increase our intuition about the curl.

The following was generated from a Jupyter notebook with the following filename and kernel.

```
notebook filename: curl-and-line-integrals.ipynb
notebook kernel: python3
```

First, load some Python packages.

```
from sympy import *
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.ticker import LogLocator
from matplotlib.colors import *
from sympy.utilities.lambdify import lambdify
from IPython.display import display, Markdown, Latex
```

Now we define some symbolic variables and functions.

```
var('x,y')
F_x = Function('F_x')(x,y)
F_y = Function('F_y')(x,y)
```

We use the same function defined in Lec. vecs.div, `quiver_plotter_2D`, to make several of these plots.

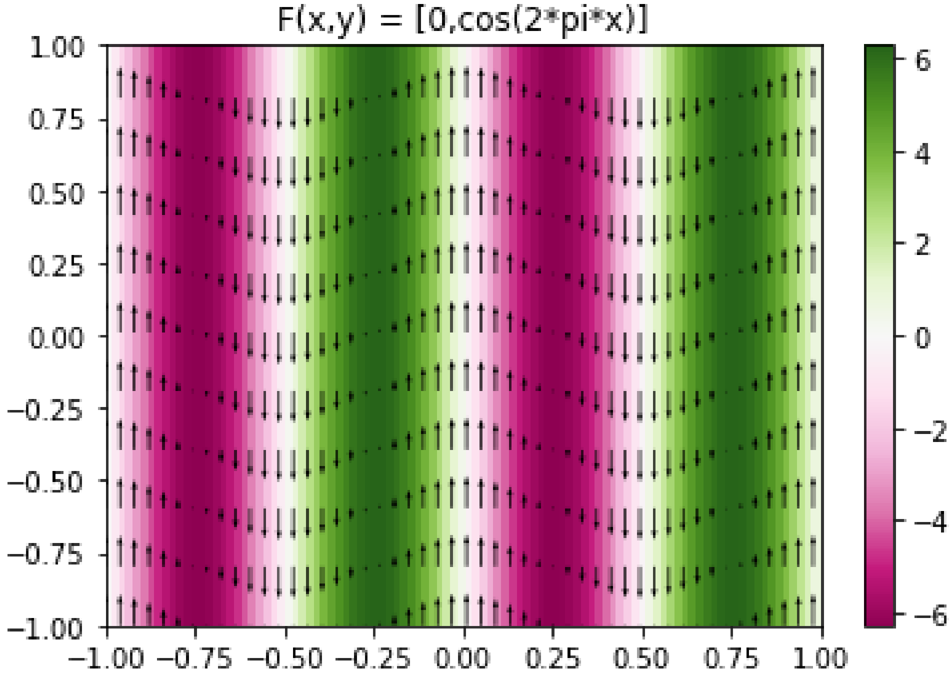

Figure curl.2: png

Let's inspect several cases.

```
p = quiver_plotter_2D(
  field={F_x:0,F_y:cos(2*pi*x)},
  density_operation='curl',
  grid_decimate_x=2,
  grid_decimate_y=10,
  grid_width=1
)
```

```
The curl is:
```

$$-2\pi \sin(2\pi x)$$

```
p = quiver_plotter_2D(
  field={F_x:0,F_y:x**2},
  density_operation='curl',
  grid_decimate_x=2,
  grid_decimate_y=20,
)
```

```
The curl is:
```

$$2x$$

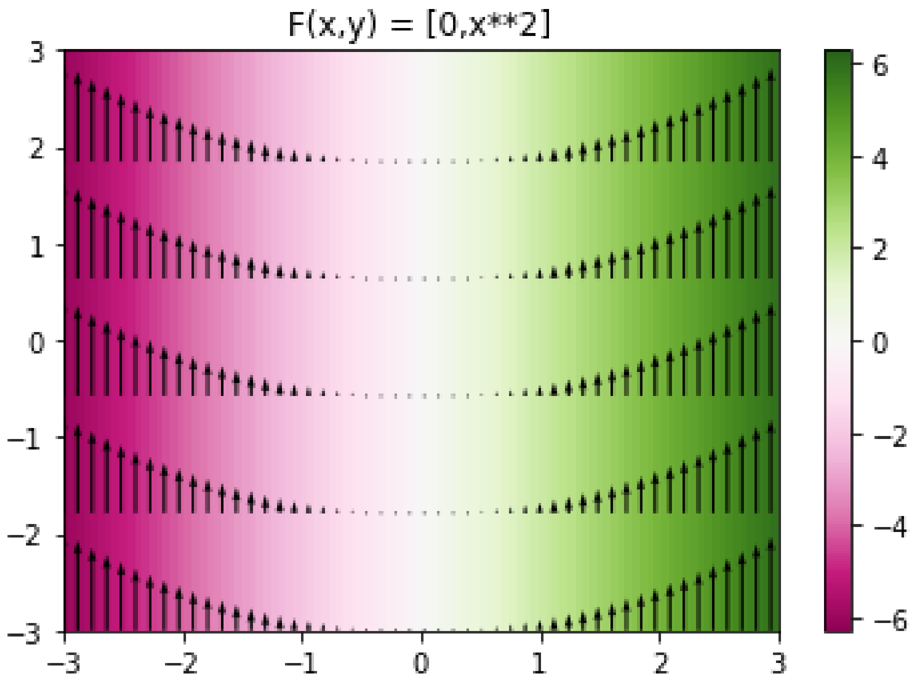

Figure curl.3: png

```
p = quiver_plotter_2D(
  field={F_x:y**2,F_y:x**2},
  density_operation='curl',
  grid_decimate_x=2,
  grid_decimate_y=20,
)
```

```
The curl is:
```

$2x - 2y$

```
p = quiver_plotter_2D(
  field={F_x:-y,F_y:x},
  density_operation='curl',
  grid_decimate_x=6,
  grid_decimate_y=6,
)
```

```
Warning: density operator is constant (no density
↪  plot)
The curl is:
```

2

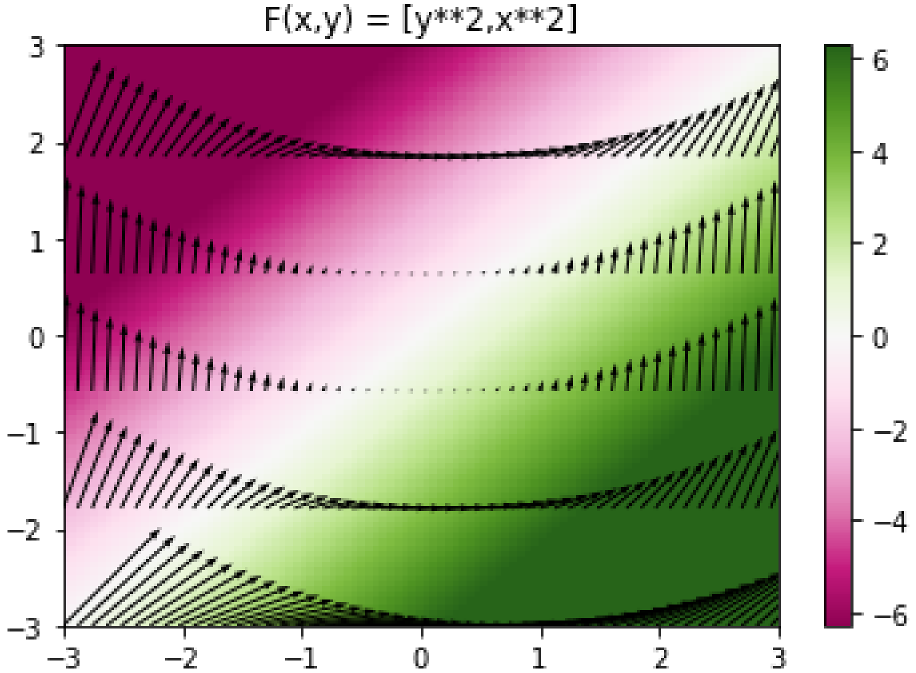

Figure curl.4: png

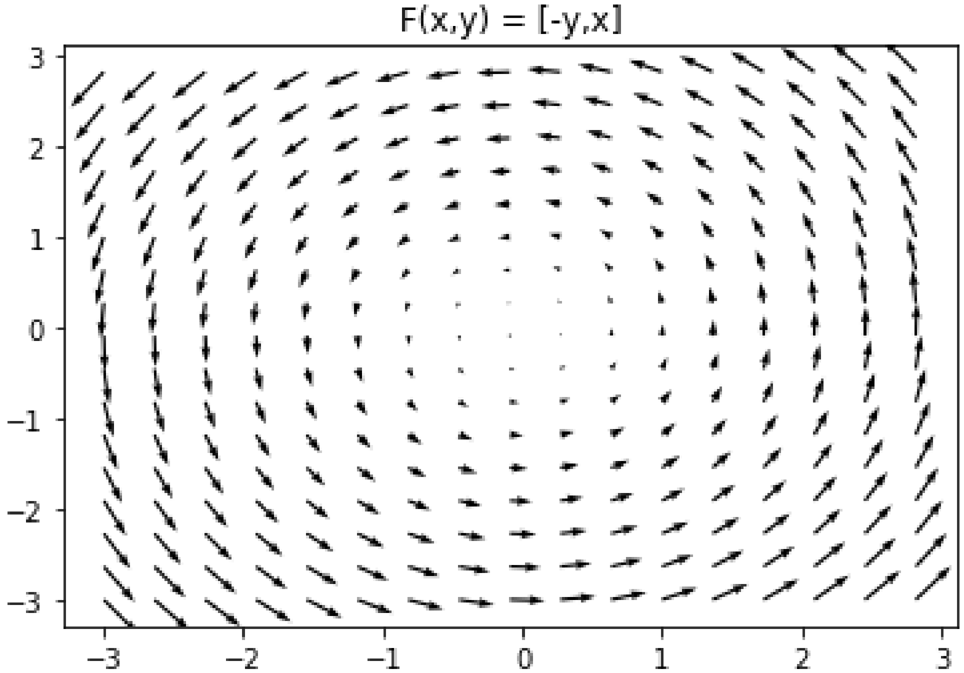

Figure curl.5: png