vecs.grad Gradient

Gradient

The **gradient** grad of a scalar-valued function $f : \mathbb{R}^3 \to \mathbb{R}$ is a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$; that is, $\operatorname{grad} f$ is a vector-valued function on $\mathbb{R}^3$. The gradient's local **direction** and **magnitude** are those of the local maximum rate of increase of f. This makes it useful in optimization (e.g. in the method of gradient descent).

In classical mechanics, quantum mechanics, relativity, string theory, thermodynamics, and continuum mechanics (and elsewhere) the **principle of least action** is taken as fundamental (Feynman, Leighton **and** Sands, 2010). This principle tells us that nature's laws quite frequently seem to be derivable by assuming a certain quantity—called *action*—is minimized. Considering, then, that the gradient supplies us with a tool for optimizing functions, it is unsurprising that the gradient enters into the equations of motion of many physical quantities.

The gradient is coordinate-independent, but its coordinate-free definitions don't add much to our intuition. In cartesian coordinates, it can be shown to be equivalent to the following.

Equation 1 gradient: cartesian coordinates

$$\operatorname{grad} f = \begin{bmatrix} \partial_x f & \partial_y f & \partial_z f \end{bmatrix}^\top$$

Vector fields from gradients are special

Although all gradients are vector fields, not all vector fields are gradients. That is, given a vector field $\mathbf{F}$, it may or may not be equal to the gradient of any scalar-valued function f. Let's say of a vector field that is a gradient that it *has*

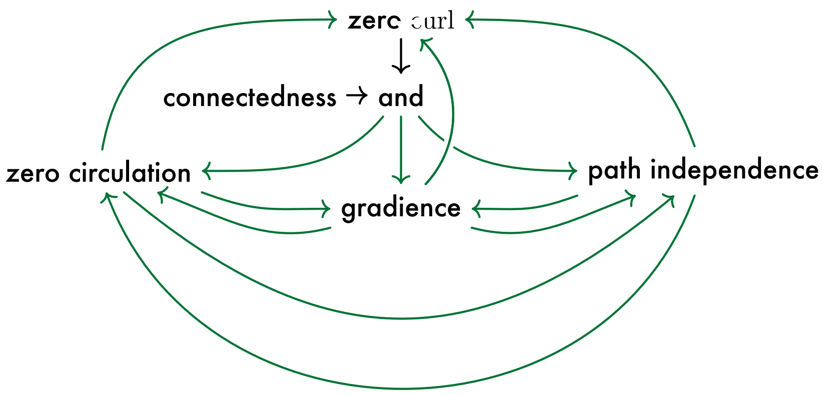

Figure grad.1: an *implication graph* relating gradience, zero curl, zero circulation, path independence, and connectedness. Green edges represent implication (a implies b) and black edges represent logical conjunctions.

gradience.[3] Those vector fields that *are* gradients have special properties. Surprisingly, those properties are connected to path independence and curl. It can be shown that iff a field is a gradient, line integrals of the field are path independent. That is, for a vector field,

$$\text{gradience} \Leftrightarrow \text{path independence}. \tag{2}$$

Considering what we know from Lec. vecs.curl about path independence we can expand Fig. curl.1 to obtain Fig. grad.1.
One implication is that *gradients have zero curl*! Many important fields that describe physical interactions (e.g. static electric fields, Newtonian gravitational fields) are gradients of scalar fields called **potentials**.

3. This is nonstandard terminology, but we're bold.

Exploring gradient

Gradient is perhaps best explored by considering it for a scalar field on $\mathbb{R}^2$. Such a field in cartesian coordinates $f(x, y)$ has gradient

$$\operatorname{grad} f = \begin{bmatrix} \partial_x f & \partial_y f \end{bmatrix}^\top \tag{3}$$

That is, $\operatorname{grad} f = \mathbf{F} = \partial_x f \,\hat{\mathbf{i}} + \partial_y f \,\hat{\mathbf{j}}$. If we overlay a quiver plot of $\mathbf{F}$ over a "color density" plot representing the f, we can increase our intuition about the gradient.
The following was generated from a Jupyter notebook with the following filename and kernel.

```
notebook filename: grad.ipynb
notebook kernel: python3
```

First, load some Python packages.

```
from sympy import *
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.ticker import LogLocator
from matplotlib.colors import *
from sympy.utilities.lambdify import lambdify
from IPython.display import display, Markdown, Latex
```

Now we define some symbolic variables and functions.

```
var('x,y')
```

```
(x, y)
```

Rather than repeat code, let's write a single function `grad_plotter_2D` to make several of these plots.
Let's inspect several cases. While considering the following plots, remember that they all have zero curl!

```
p = grad_plotter_2D(
  field=x,
)
```

```
The gradient is:
```

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

```
p = grad_plotter_2D(
  field=x+y,
)
```

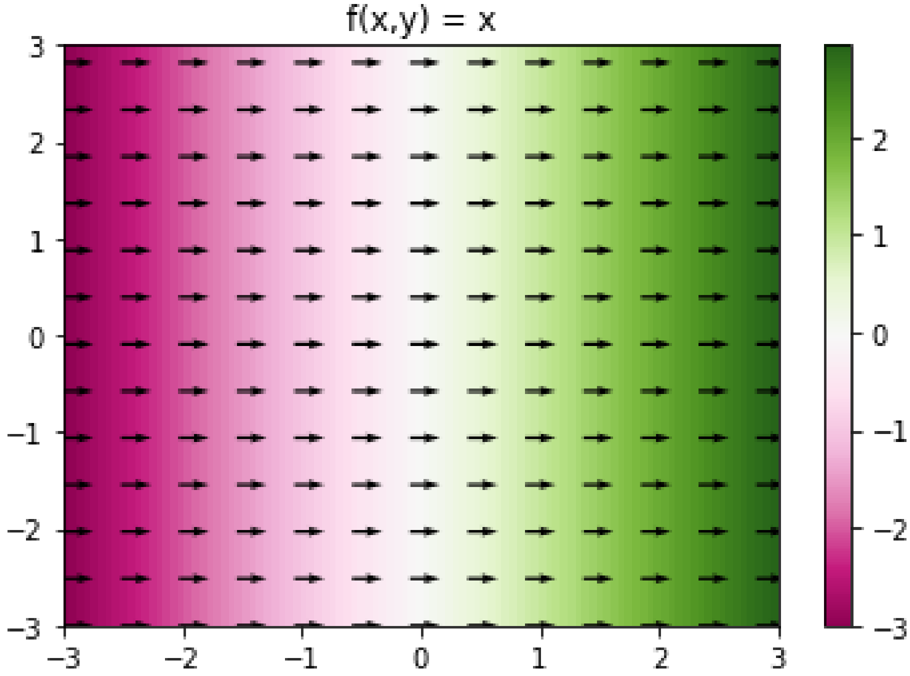

Figure grad.2: png

```
The gradient is:
```

$$\begin{bmatrix} 1 & 1 \end{bmatrix}$$

```
p = grad_plotter_2D(
  field=1,
)
```

```
Warning: field is constant (no plot)
The gradient is:
```

$$\begin{bmatrix} 0 & 0 \end{bmatrix}$$

Gravitational potential

Gravitational potentials have the form of 1/distance. Let's check out the gradient.

```
p = grad_plotter_2D(
  field=1/sqrt(x**2+y**2),
  norm=SymLogNorm(linthresh=.3, linscale=.3),
  mask=True,
)
```

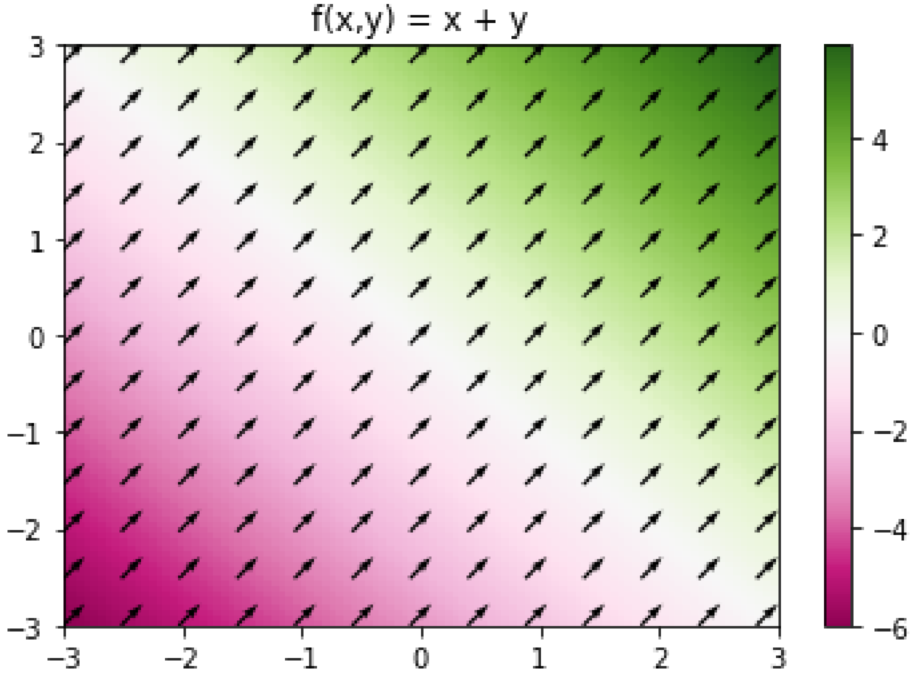

Figure grad.3: png

```
The gradient is:
```

$$\left[-\frac{x}{(x^2+y^2)^{\frac{3}{2}}} \quad -\frac{y}{(x^2+y^2)^{\frac{3}{2}}} \right]$$

Conic section fields

Gradients of **conic section** fields can be explored.
The following is called a **parabolic field**.

```
p = grad_plotter_2D(
  field=x**2,
)
```

```
The gradient is:
```

$$\begin{bmatrix} 2x & 0 \end{bmatrix}$$

The following are called **eliptic fields**.

```
p = grad_plotter_2D(
  field=x**2+y**2,
)
```

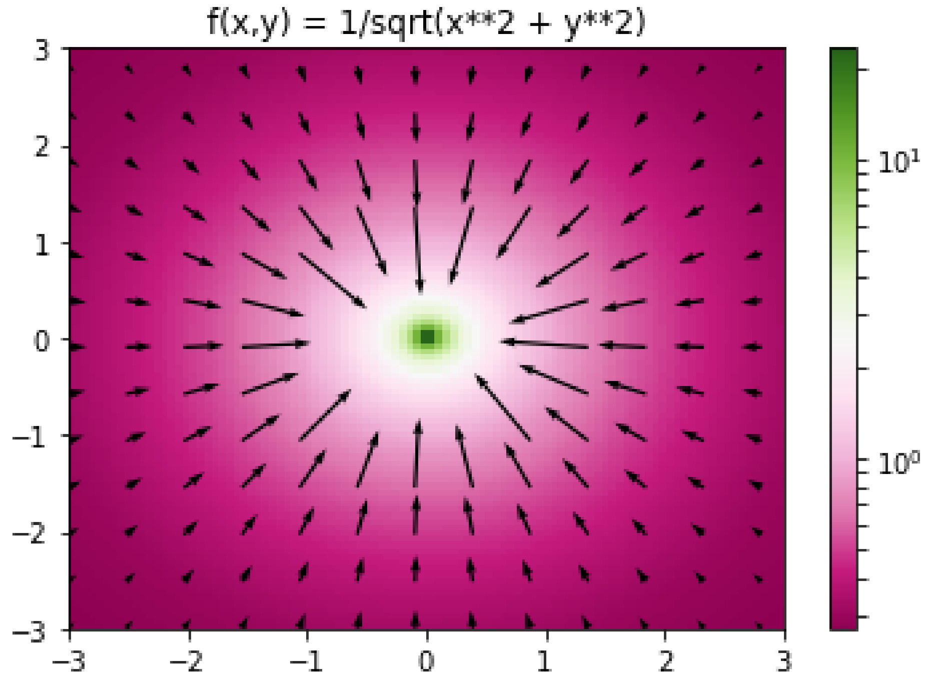

Figure grad.4: png

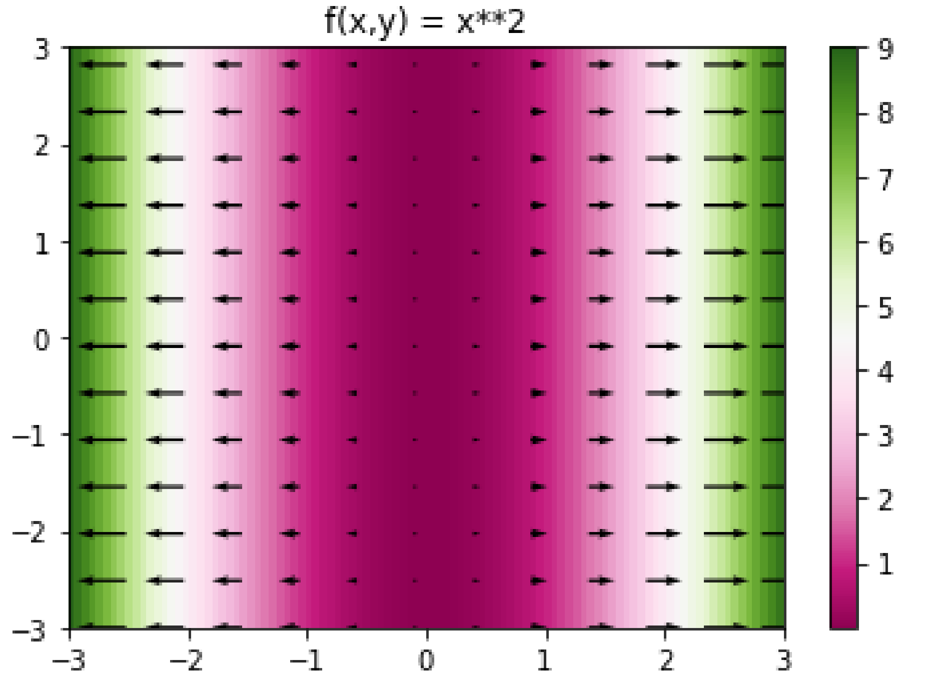

Figure grad.5: png

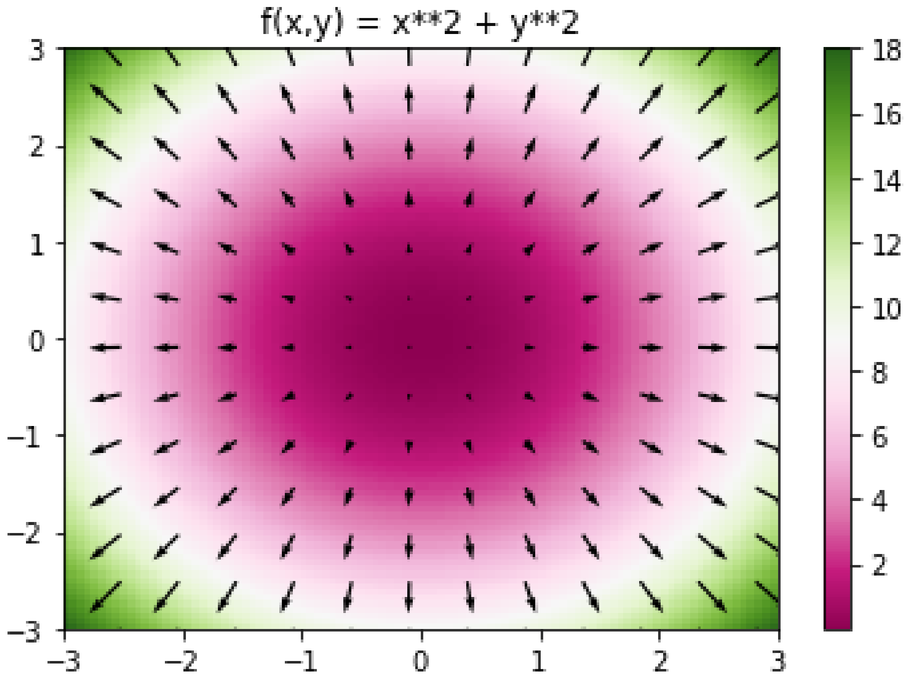

Figure grad.6: png

```
The gradient is:
```

$$\begin{bmatrix} 2x & 2y \end{bmatrix}$$

```
p = grad_plotter_2D(
  field=-x**2-y**2,
)
```

```
The gradient is:
```

$$\begin{bmatrix} -2x & -2y \end{bmatrix}$$

The following is called a **hyperbolic field**.

```
p = grad_plotter_2D(
  field=x**2-y**2,
)
```

```
The gradient is:
```

$$\begin{bmatrix} 2x & -2y \end{bmatrix}$$

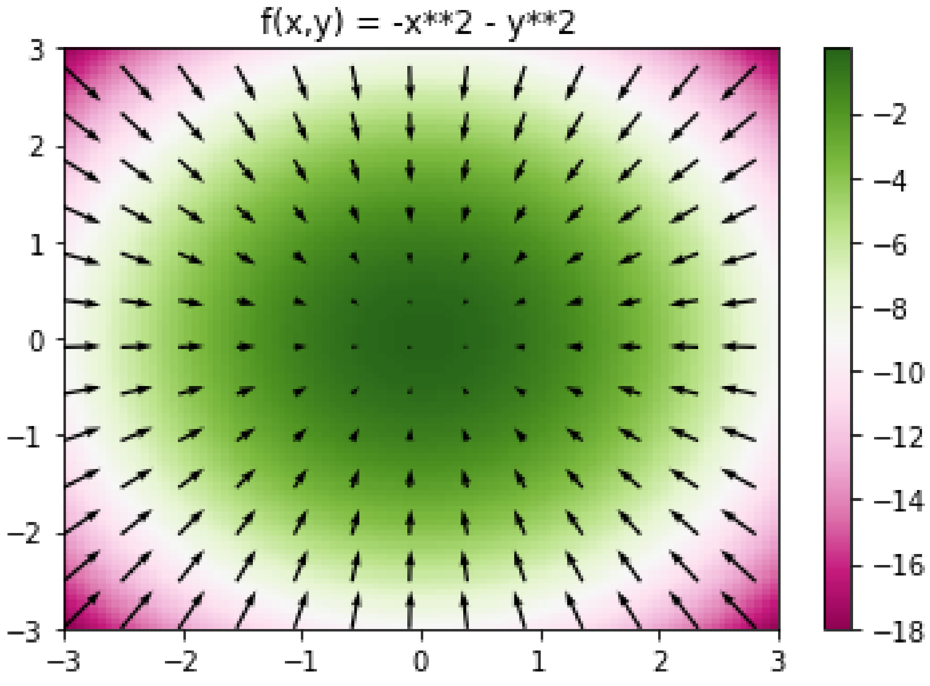

Figure grad.7: png

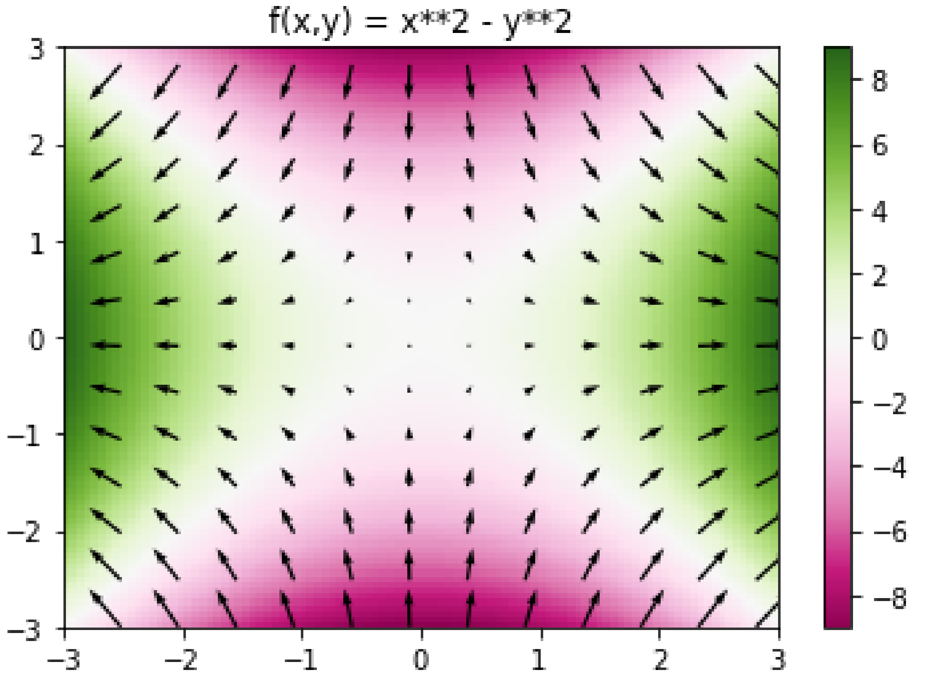

Figure grad.8: png