vecs.stoked Stokes and divergence theorems

Two theorems allow us to exchange certain integrals in $\mathbb{R}^3$ for others that are easier to evaluate.

The divergence theorem

The **divergence theorem** asserts the equality of the surface integral of a vector field $\mathbf{F}$ and the **triple integral** of $\operatorname{div}\mathbf{F}$ over the volume V enclosed by the surface S in $\mathbb{R}^3$. That is,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV. \tag{1}$$

Caveats are that V is a closed region bounded by the **orientable**[4] surface S and that $\mathbf{F}$ is continuous and continuously differentiable over a region containing V. This theorem makes some intuitive sense: we can think of the divergence inside the volume "accumulating" via the triple integration and equaling the corresponding surface integral. For more on the divergence theorem, see Kreyszig (2011, § 10.7) and Schey (2005, pp. 45-52).

4. A surface is orientable if a consistent normal direction can be defined. Most surfaces are orientable, but some, notably the Möbius strip, cannot be. See Kreyszig (2011, § 10.6) for more.

A lovely application of the divergence theorem is that, for any quantity of conserved stuff (mass, charge, spin, etc.) distributed in a spatial $\mathbb{R}^3$ with time-dependent density $\rho : \mathbb{R}^4 \to \mathbb{R}$ and velocity field $\boldsymbol{v} : \mathbb{R}^4 \to \mathbb{R}^3$, the divergence theorem can be applied to find that

$$\partial_t \rho = -\operatorname{div}(\rho\boldsymbol{v}), \tag{2}$$

which is a more general form of a **continuity equation**, one of the governing equations of many physical phenomena. For a derivation of this equation, see Schey (**ibidem**, pp. 49-52).

The Kelvin-Stokes' theorem

The **Kelvin-Stokes' theorem** asserts the equality of the circulation of a vector field $\mathbf{F}$ over a closed curve C and the surface integral of $\operatorname{curl} \mathbf{F}$ over a surface S that has boundary C. That is, for $\mathbf{r}(t)$ a parameterization of C and surface normal $\mathbf{n}$,

$$\oint_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \iint_S \mathbf{n} \cdot \operatorname{curl} \mathbf{F}\,dS. \tag{3}$$

Caveats are that S is **piecewise smooth**,[5] its boundary C is a piecewise smooth simple closed curve, and $\mathbf{F}$ is continuous and continuously differentiable over a region containing S. This theorem is also somewhat intuitive: we can think of the divergence over the surface "accumulating" via the surface integration and equaling the corresponding circulation. For more on the Kelvin-Stokes' theorem, see Kreyszig (2011, § 10.9) and Schey (2005, pp. 93-102).

5. A surface is *smooth* if its normal is continuous everywhere. It is *piecewise smooth* if it is composed of a finite number of smooth surfaces.

Related theorems

Greene's theorem is a two-dimensional special case of the Kelvin-Stokes' theorem. It is described by Kreyszig (2011, § 10.9).
It turns out that all of the above theorems (and the fundamental theorem of calculus, which relates the derivative and integral) are special cases of the **generalized Stokes' theorem** defined by differential geometry. We would need a deeper understanding of differential geometry to understand this theorem. For more, see Lee (2012, Ch. 16).