four.series   Fourier series

1   Fourier series are mathematical series that can represent a periodic signal as a sum of sinusoids at different amplitudes and frequencies. They are useful for solving for the response of a system to periodic inputs. However, they are probably most important *conceptually*: they are our gateway to thinking of signals in the **frequency domain**—that is, as functions of *frequency* (not time). To represent a function as a Fourier series is to **analyze** it as a sum of sinusoids at different frequencies[1] ω_n and amplitudes a_n. Its **frequency spectrum** is the functional representation of amplitudes a_n versus frequency ω_n.

1. It's important to note that the symbol ω_n, in this context, is not the natural frequency, but a frequency indexed by integer n.

2   Let's begin with the definition.

Definition four.1: Fourier series: trigonometric form

The *Fourier analysis* of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} y(t)dt \tag{1}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t)\cos(\omega_n t)dt \tag{2}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} y(t)\sin(\omega_n t)dt. \tag{3}$$

The *Fourier synthesis* of a periodic function $y(t)$ with analysis components a_n and b_n corresponding to ω_n is

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \sin(\omega_n t). \tag{4}$$

3   Let's consider the complex form of the Fourier series, which is analogous to Definition four.1. It may be helpful to review

Euler's formula(s) – see Appendix D.01.

Definition four.2: Fourier series: complex form

The *Fourier analysis* of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$, period T, and angular frequency $\omega_n = 2\pi n/T$,

$$c_{\pm n} = \frac{1}{T} \int_{-T/2}^{T/2} y(t) e^{-j\omega_n t} dt. \qquad (5)$$

The *Fourier synthesis* of a periodic function $y(t)$ with analysis components c_n corresponding to ω_n is

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}. \qquad (6)$$

4 We call the integer n a **harmonic** and the frequency associated with it,

$$\omega_n = 2\pi n/T, \qquad (7)$$

the **harmonic frequency**. There is a special name for the first harmonic ($n = 1$): the **fundamental frequency**. It is called this because all other frequency components are integer multiples of it.

5 It is also possible to convert between the two representations above.

Definition four.3: Fourier series: converting between forms

The complex Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and a_n and b_n as defined above,

$$c_{\pm n} = \frac{1}{2} \left(a_{|n|} \mp j b_{|n|} \right) \qquad (8)$$

The sinusoidal Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and c_n as defined above,

$$a_n = c_n + c_{-n} \text{ and} \qquad (9)$$

$$b_n = j \left(c_n - c_{-n} \right). \qquad (10)$$

6   The **harmonic amplitude** C_n is

$$C_n = \sqrt{a_n^2 + b_n^2} \qquad (11)$$

$$= 2\sqrt{c_n c_{-n}}. \qquad (12)$$

A **magnitude line spectrum** is a graph of the harmonic amplitudes as a function of the harmonic frequencies. The **harmonic phase** is

$$\theta_n = -\arctan_2(b_n, a_n) \quad \text{(see Appendix C.02)}$$

$$= \arctan_2(\text{Im}(c_n), \text{Re}(c_n)). \qquad (13)$$

7   The illustration of Fig. series.1 shows how sinusoidal components sum to represent a square wave. A line spectrum is also shown.
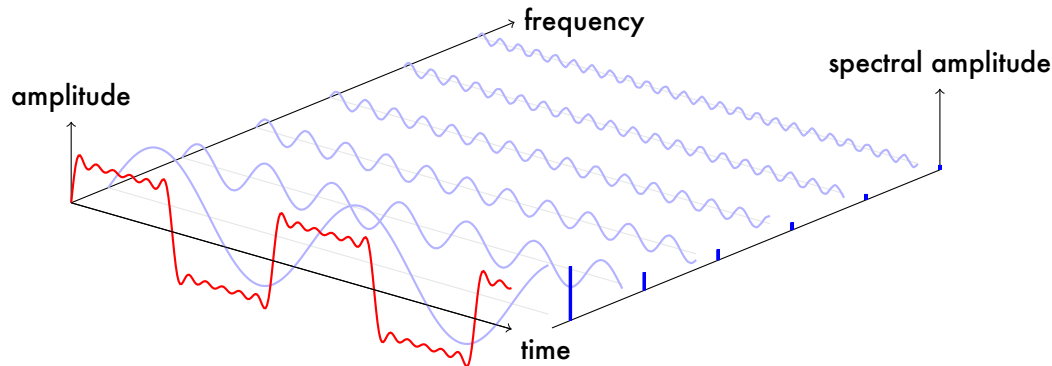

Figure series.1: a partial sum of Fourier components of a square wave shown through time and frequency. The spectral amplitude shows the amplitude of the corresponding Fourier component.

8   Let us compute the associated spectral components in the following example.

Example four.series-1   **re: Fourier series analysis: line spectrum**

Compute the first five harmonic amplitudes that represent the line spectrum for a square wave in the figure above.