pde.sturm Sturm-liouville problems

Before we introduce an important solution method for PDEs in Lec. pde.separation, we consider an *ordinary* differential equation that will arise in that method when dealing with a single spatial dimension x: the **sturm-liouville (S-L) differential equation**. Let p, q, σ be functions of x on open interval (a, b). Let X be the dependent variable and λ constant. The **regular S-L problem** is the S-L ODE[2]

$$\frac{d}{dx}\left(pX'\right) + qX + \lambda\sigma X = 0 \tag{1a}$$

with boundary conditions

$$\beta_1 X(a) + \beta_2 X'(a) = 0 \tag{2a}$$

$$\beta_3 X(b) + \beta_4 X'(b) = 0 \tag{2b}$$

with coefficients $\beta_i \in \mathbb{R}$. This is a type of **boundary value problem**.

This problem has nontrivial solutions, called **eigenfunctions** $X_n(x)$ with $n \in \mathbb{Z}_+$, corresponding to specific values of $\lambda = \lambda_n$ called **eigenvalues**.[3] There are several important theorems proven about this (see Haberman (2018, § 5.3)). Of greatest interest to us are that

1. there exist an infinite number of eigenfunctions X_n (unique within a multiplicative constant),
2. there exists a unique corresponding *real* eigenvalue λ_n for each eigenfunction X_n,
3. the eigenvalues can be ordered as $\lambda_1 < \lambda_2 < \cdots$,
4. eigenfunction X_n has $n - 1$ zeros on open interval (a, b),
5. the eigenfunctions X_n form an orthogonal basis with respect to weighting function σ such that any piecewise continuous function $f : [a, b] \to \mathbb{R}$ can be represented by a generalized fourier series on $[a, b]$.

2. For the S-L problem to be *regular*, it has the additional constraints that p, q, σ are continuous and $p, \sigma > 0$ on $[a, b]$. This is also sometimes called the sturm-liouville eigenvalue problem. See Haberman (2018, § 5.3) for the more general (non-regular) S-L problem and Haberman (**ibidem**, § 7.4) for the multi-dimensional analog.

3. These eigenvalues are closely related to, but distinct from, the "eigenvalues" that arise in systems of linear ODEs.

This last theorem will be of particular interest in Lec. pde.separation.

Types of boundary conditions

Boundary conditions of the sturm-liouville kind (2) have four sub-types:

dirichlet for just $\beta_2, \beta_4 = 0$,
neumann for just $\beta_1, \beta_3 = 0$,
robin for all $\beta_i \neq 0$, and
mixed if $\beta_1 = 0$, $\beta_3 \neq 0$; if $\beta_2 = 0$, $\beta_4 \neq 0$.

There are many problems that are *not* regular sturm-liouville problems. For instance, the right-hand sides of Eq. 2 are zero, making them **homogeneous boundary conditions**; however, these can also be nonzero. Another case is **periodic boundary conditions**:

$$X(a) = X(b) \tag{3a}$$
$$X'(a) = X'(b). \tag{3b}$$

Example pde.sturm-1 re: a sturm-liouville problem with dirichlet boundary conditions

Consider the differential equation

$$X'' + \lambda X = 0 \tag{4}$$

with dirichlet boundary conditions on the boundary of the interval $[0, L]$

$$X(0) = 0 \quad \text{and} \quad X(L) = 0. \tag{5}$$

Solve for the eigenvalues and eigenfunctions.

This is a sturm-liouville problem, so we know the eigenvalues are real. The well-known general solutions to the ODE is

$$X(x) = \begin{cases} k_1 + k_2 x & \lambda = 0 \\ k_1 e^{j\sqrt{\lambda}x} + k_2 e^{-j\sqrt{\lambda}x} & \text{otherwise} \end{cases} \tag{6}$$

with real constants k_1, k_2. The solution must also satisfy the boundary conditions. Let's apply them to the case of $\lambda = 0$ first:

$$X(0) = 0 \Rightarrow k_1 + k_2(0) = 0 \Rightarrow k_1 = 0 \tag{7}$$

$$X(L) = 0 \Rightarrow k_1 + k_2(L) = 0 \Rightarrow k_2 = -k_1/L. \tag{8}$$

Together, these imply $k_1 = k_2 = 0$, which gives the *trivial solution* $X(x) = 0$, in which we aren't interested. We say, then, for nontrivial solutions $\lambda \neq 0$. Now let's check $\lambda < 0$. The solution becomes

$$X(x) = k_1 e^{-\sqrt{|\lambda|}x} + k_2 e^{\sqrt{|\lambda|}x} \tag{9}$$

$$= k_3 \cosh(\sqrt{|\lambda|}x) + k_4 \sinh(\sqrt{|\lambda|}x) \tag{10}$$

where k_3 and k_4 are real constants. Again applying the boundary conditions:

$$X(0) = 0 \Rightarrow k_3 \cosh(0) + k_4 \sinh(0) = 0 \Rightarrow k_3 + 0 = 0 \Rightarrow k_3 = 0$$

$$X(L) = 0 \Rightarrow 0 \cosh(\sqrt{|\lambda|}L) + k_4 \sinh(\sqrt{|\lambda|}L) = 0 \Rightarrow k_4 \sinh(\sqrt{|\lambda|}L) = 0.$$

However, $\sinh(\sqrt{|\lambda|}L) \neq 0$ for $L > 0$, so $k_4 = k_3 = 0$—again, the trivial solution. Now let's try $\lambda > 0$. The solution can be written

$$X(x) = k_5 \cos(\sqrt{\lambda}x) + k_6 \sin(\sqrt{\lambda}x). \tag{11}$$

Applying the boundary conditions for this case:

$$X(0) = 0 \Rightarrow k_5 \cos(0) + k_6 \sin(0) = 0 \Rightarrow k_5 + 0 = 0 \Rightarrow k_5 = 0$$

$$X(L) = 0 \Rightarrow 0 \cos(\sqrt{\lambda}L) + k_6 \sin(\sqrt{\lambda}L) = 0 \Rightarrow k_6 \sin(\sqrt{\lambda}L) = 0.$$

Now, $\sin(\sqrt{\lambda}L) = 0$ for

$$\sqrt{\lambda}L = n\pi \Rightarrow$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2. \qquad (n \in \mathbb{Z}_+)$$

Therefore, the only nontrivial solutions that satisfy both the ODE and the boundary conditions are the *eigenfunctions*

$$X_n(x) = \sin\left(\sqrt{\lambda_n}x\right) \tag{12a}$$

$$= \sin\left(\frac{n\pi}{L}x\right) \tag{12b}$$

with corresponding *eigenvalues*

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2. \tag{13}$$

Note that because $\lambda > 0$, λ_1 is the lowest eigenvalue.

Plotting the eigenfunctions

The following was generated from a Jupyter notebook with the following filename and kernel.

```
notebook filename: eigenfunctions_example_plot.ipynb
notebook kernel: python3
```

First, load some Python packages.

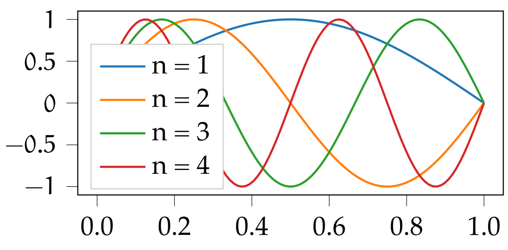
```
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Markdown, Latex
```

Set $L = 1$ and compute values for the first four eigenvalues `lambda_n` and eigenfunctions `X_n`.

```
L = 1
x = np.linspace(0,L,100)
n = np.linspace(1,4,4,dtype=int)
lambda_n = (n*np.pi/L)**2
X_n = np.zeros([len(n),len(x)])
for i,n_i in enumerate(n):
  X_n[i,:] = np.sin(np.sqrt(lambda_n[i])*x)
```

Plot the eigenfunctions.

```
for i,n_i in enumerate(n):
  plt.plot(
    x,X_n[i,:],
    linewidth=2,label='n = '+str(n_i)
  )
plt.legend()
plt.show() # display the plot
```



We see that the fourth of the S-L theorems appears true: $n - 1$ zeros of X_n exist on the open interval $(0, 1)$.