pde.exe Exercises for Chapter pde

Exercise pde.horticulture

______/20 p.

The PDE of Example pde.separation-1 can be used to describe the conduction of heat along a long, thin rod, insulated along its length, where $u(t, x)$ represents temperature. The initial and dirichlet boundary conditions in that example would be interpreted as an initial temperature distribution along the bar and fixed temperatures of the ends. Now consider the same PDE

$$\partial_t u(t, x) = k \partial_{xx}^2 u(t, x) \tag{1}$$

with real constant k, with mixed boundary conditions on interval $x \in [0, L]$

$$u(t, 0) = 0 \tag{2a}$$

$$\partial_x u(t, x)|_{x=L} = 0, \tag{2b}$$

and with initial condition

$$u(0, x) = f(x), \tag{3}$$

where f is some piecewise continuous function on $[0, L]$. This represents the insulation of one end (L) of the rod and the other end (0) is held at a fixed temperature.

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions X_n and eigenvalues λ_n.
c. Solve for the general solution of the time variable ODE.
d. Write the product solution and apply the initial condition $f(x)$ by constructing it from a generalized fourier series of the product solution.
e. Let $L = k = 1$ and

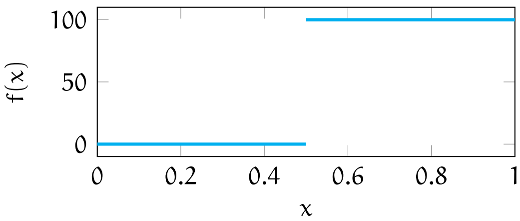

Figure exe.1: initial condition for Exercise pde..

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, L/2) \\ 100 & \text{for } x \in [L/2, L] \end{cases} \tag{4}$$

as shown in Fig. exe.1. Compute the solution series components. Plot the sum of the first 50 terms over x and t.

Exercise pde.poltergeist

The PDE of Example pde.separation-1 can be used to describe the conduction of heat along a long, thin rod, insulated along its length, where $u(t, x)$ represents temperature. The initial and dirichlet boundary conditions in that example would be interpreted as an initial temperature distribution along the bar and fixed temperatures of the ends. Now consider the same PDE

$$\partial_t u(t, x) = k\partial^2_{xx} u(t, x) \tag{5}$$

with real constant k, now with *neumann* boundary conditions on interval $x \in [0, L]$

$$\partial_x u|_{x=0} = 0 \quad \text{and} \quad \partial_x u|_{x=L} = 0, \tag{6a}$$

and with initial condition

$$u(0, x) = f(x), \tag{7}$$

where f is some piecewise continuous function on $[0, L]$. This represents the complete insulation of the ends of the rod, such that no heat flows from the ends (or from anywhere else).

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions X_n and eigenvalues λ_n.
c. Solve for the general solution of the time variable ODE.

d. Write the product solution and apply the initial condition $f(x)$ by constructing it from a generalized fourier series of the product solution.
e. Let $L = k = 1$ and $f(x) = 100 - 200/L\,|x - L/2|$ as shown in Fig. exe.2. Compute the solution series components. Plot the sum of the first 50 terms over x and t.

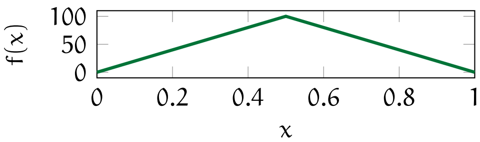

Figure exe.2: initial condition for ??.

Exercise pde.kathmandu

Consider the free vibration of a uniform and relatively thin beam—with modulus of elasticity E, second moment of cross-sectional area I, and mass-per-length μ—pinned at each end. The PDE describing this is a version of the euler-bernoulli beam equation for vertical motion u:

$$\partial^2_{tt} u(t, x) = -\alpha^2 \partial^4_{xxxx} u(t, x) \tag{8}$$

with real constant α defined as

$$\alpha^2 = \frac{EI}{\mu}. \tag{9}$$

Pinned supports fix vertical motion such that we have boundary conditions on interval $x \in [0, L]$

$$u(t, 0) = 0 \quad \text{and} \quad u(t, L) = 0. \tag{10a}$$

Additionally, pinned supports cannot provide a moment, so

$$\partial^2_{xx} u|_{x=0} = 0 \quad \text{and} \quad \partial^2_{xx} u|_{x=L} = 0. \tag{10b}$$

Furthermore, consider the initial conditions

$$u(0,x) = f(x), \quad \text{and} \quad \partial_t u|_{t=0} = 0. \tag{11a}$$

where f is some piecewise continuous function on $[0, L]$.

a. Assume a product solution, separate variables into $X(x)$ and $T(t)$, and set the separation constant to $-\lambda$.
b. Solve the boundary value problem for its eigenfunctions X_n and eigenvalues λ_n. Assume real $\lambda > 0$ (it's true but tedious to show).
c. Solve for the general solution of the time variable ODE.
d. Write the product solution and apply the initial conditions by constructing it from a generalized fourier series of the product solution.
e. Let $L = \alpha = 1$ and $f(x) = \sin(10\pi x/L)$ as shown in Fig. exe.3. Compute the solution series components. Plot the sum of the first 50 terms over x and t.

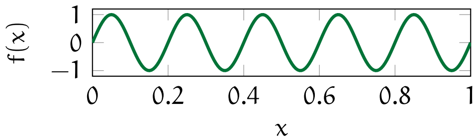

Figure exe.3: initial condition for Exercise pde..

Exercise pde.hurried

Given the 1D heat equation, ______/40 p.

$$\frac{\partial}{\partial t} u(t,x) = \alpha \frac{\partial^2}{\partial x^2} u(t,x),$$

with boundary conditions,

$$\frac{\partial}{\partial x} u(t,x)\Big|_{x=L} = 0$$
$$u(t,0) = 0,$$

and initial condition,

$$u(0,x) = \begin{cases} 1 & \frac{L}{3} \leqslant x \leqslant \frac{2L}{3} \\ 0 & \text{otherwise} \end{cases}$$

1. show that this PDE is separable,
2. solve the sturm-liouville boundary condition problem,
3. find the fourier coefficients, and
4. given $L = 1$, $\alpha = 1$, and using the first 100 terms of the infinite sum, plot the solution at $t = 0$, $t = 0.01$, and $t = 0.1$.

Optimization