Lecture 03.09 Response to periodic inputs

We have already considered the response of first- and second-order measurement systems to sinusoidal inputs (measurands). These are not the only periodic inputs encountered by measurement systems; in fact, we frequently encounter non-sinusoidal periodic inputs.

Fortunately, we already have the mathematical apparatus to deal with these inputs. Recall that a periodic signal u with period T has a *Fourier series* representation, for $n \in \mathbb{N}_0$ and $\omega_n \equiv 2\pi n/T$ is the angular frequency of component n,

Fourier series

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} C_n \sin(\omega_n t + \phi_n). \tag{03.43}$$

where we have written the sum in terms of harmonic amplitudes C_n and phases ϕ_n defined as via A.02.11:

$$C_n = \sqrt{a_n^2 + b_n^2} \text{ and} \tag{03.44}$$

$$\phi_n = \arctan \frac{b_n}{a_n} \tag{03.45}$$

and where a_n and b_n are found from the trigonometric Fourier series analysis

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} u(t) \cos(\omega_n t)\, dt \tag{03.46}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} u(t) \sin(\omega_n t)\, dt. \tag{03.47}$$

In other words, *a periodic signal can be represented as a sum of sinusoids.*

When we combine this with the *principle of superposition*—specifically with the fact that, for linear systems, *the linear combination of inputs yields an equivalent linear combination of outputs*—we can compute the response of a system to a periodic input by

principle of superposition

1. representing the input u with a Fourier series,
2. computing the response of the system to each term in the series, and
3. summing the result.

This is valid for transient and steady-state analysis, but, when working with periodic functions, we typically are most concerned with steady-state.

Conveniently, the steady-state response y_n of a system with frequency response function $H(j\omega)$ to sinusoidal forcing $u_n = C_n \sin(\omega_n t + \phi_n)$ has already been developed:

The special case is y_0, which is[2]

From the principle of superposition, the output to the sum of the inputs u_n is just the sum of outputs y_n:

In Example 02.02-1, we found that a square wave of amplitude one has trignometric Fourier series components

$$\begin{aligned} a_n &= 0 \text{ and} \\ b_n &= \frac{2}{n\pi}(1 - \cos(n\pi)) \\ &= \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd.} \end{cases} \end{aligned}$$

Therefore, from the definitions of C_n and ϕ_n, with $b_n \geqslant 0$,

$$\begin{aligned} C_n &= b_n \text{ and} \\ \phi_n &= \arctan \frac{b_n}{a_n} \\ &= \begin{cases} ¿ & \text{for } n \text{ even} \\ \pi/2 & \text{for } n \text{ odd.} \end{cases} \end{aligned}$$

If we consider the steady-state response of a system with frequency response function $H(j\omega)$ to this square wave input, we can create Figure 03.12 and Figure 03.13, showing how the system responds to this input. These figures are generated by applying the expression for y_n.

[2]This is derived by assuming an input amplitude $a_0/2$ and angular frequency 0 rad/s.

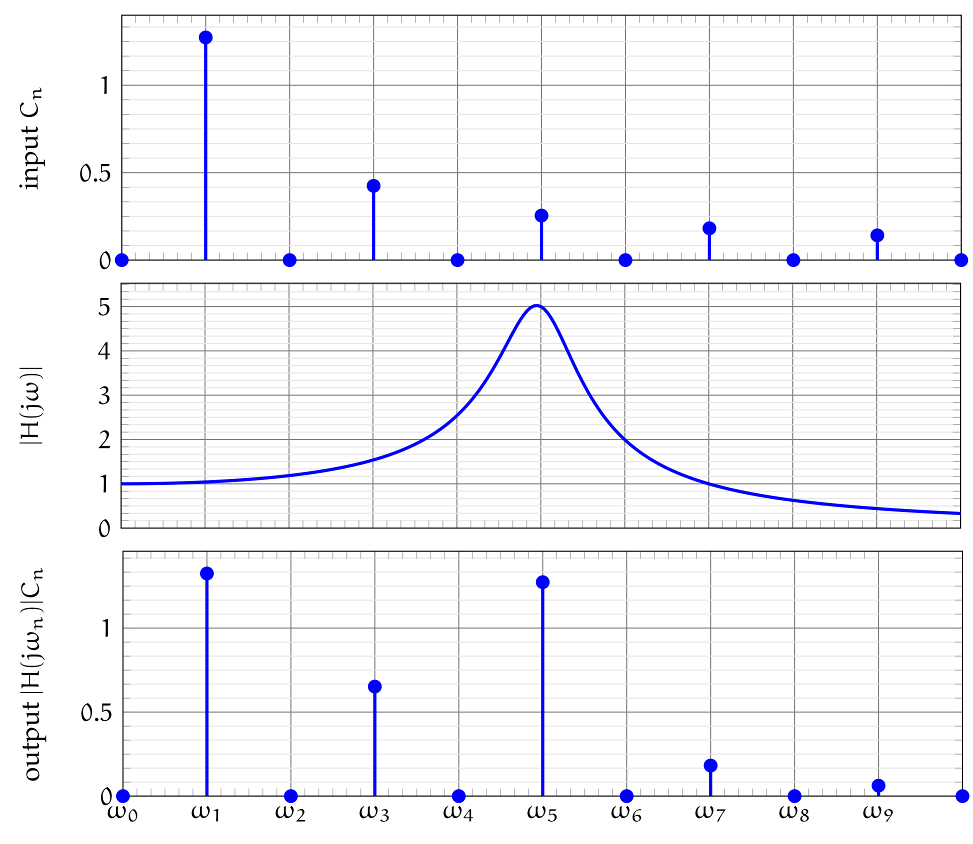

Figure 03.12: the magnitude line spectrum C_n of the input, which is operated on by the measurement system with frequency response function $H(j\omega)$ to form the output magnitude line spectrum $|H(j\omega_n)|C_n$.

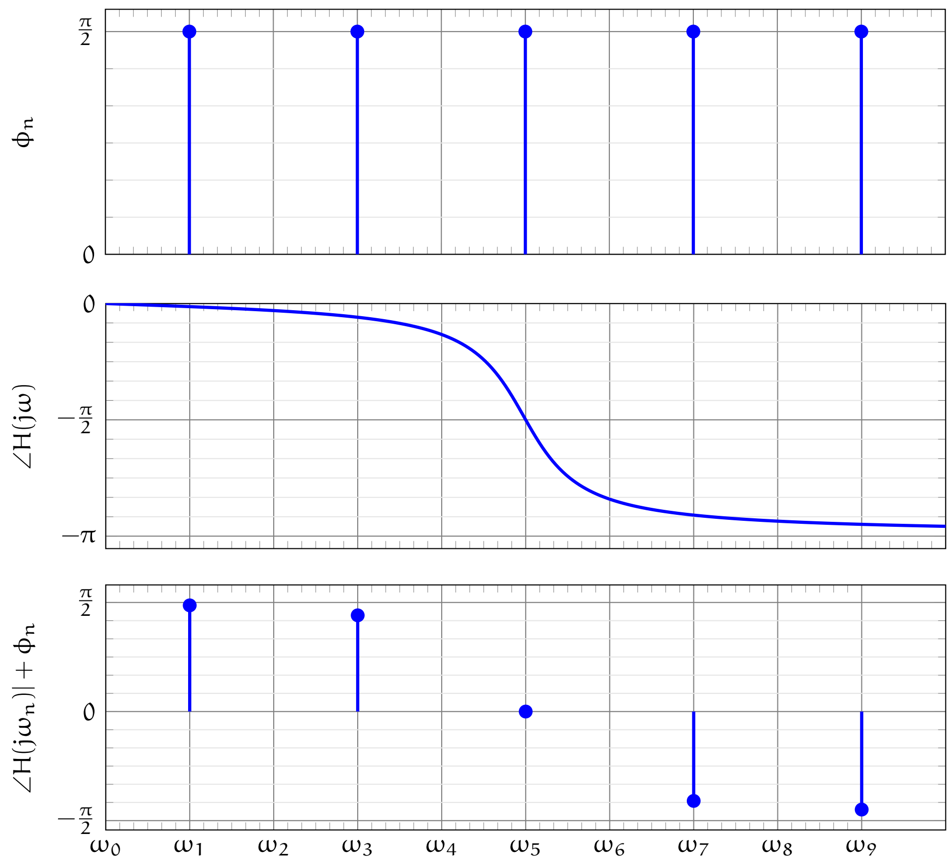

Figure 03.13: the phase line spectrum ϕ_n of the input, which is operated on by the measurement system with frequency response function $H(j\omega)$ to form the output phase line spectrum $\angle H(j\omega_n) + \phi_n$.