Classical solution of linear differential equations

Recall that we can rewrite a SISO system's state equations

$$\frac{d\vec{x}}{dt} = A\vec{x} + Bu$$
$$y = C\vec{x} + Du$$

in classical SISO form

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u.$$

The right-hand side is a known forcing function

$$f(t) \equiv b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u \quad .$$

It is convenient to rewrite the classical ODE as

$$\boxed{\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = f(t)} \quad . \qquad (*)$$

The existence and uniqueness theorem (see R+W p. 252) tells us that:

1. given initial conditions $y(t_0)$, $\left.\frac{dy}{dt}\right|_{t=t_0}$, $\ldots$, $\left.\frac{d^{n-1}y}{dt^{n-1}}\right|_{t=t_0}$, and $f(t)$ for $t \geq t_0$,
   there exists a solution; and
2. this solution is unique, meaning it is the only one.

The general solution (or total solution) $y(t)$ is the sum of the homogeneous solution $y_h(t)$ and the particular solution $y_p(t)$:

$$\boxed{y(t) = y_h(t) + y_p(t)} \quad .$$

We will now examine each of these, in turn.

Homogeneous solution

The homogeneous solution is defined as the solution of (*) with $f(t)=0$:

$$\frac{d^n y_h}{dt^n} + a_{n-1}\frac{d^{n-1} y_h}{dt^{n-1}} + \cdots + a_1 \frac{dy_h}{dt} + a_0 y_h = 0 \quad . \quad (**)$$

The standard method of solving (**) is to assume a solution: $y_h(t) = Ce^{\lambda t}$, where $C \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. It can be easily verified as a solution for nonzero constants C_i and all solutions of the characteristic equation

$$\boxed{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0} \quad .$$

The left-hand side is called the characteristic polynomial. It has n roots λ_i and, therefore, n solution $C_i e^{\lambda_i t}$. Any linear combination of these is also a solution to (**).

If there are n distinct roots, in general $\boxed{y_h(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t}}$.

If a given root is repeated, e.g. $\lambda_i = \lambda_j = \lambda$, the homogeneous solution above doesn't represent all possible solutions. This solution can be supplemented by extra terms. If a root λ has multiplicity m, the m terms in the homogeneous solution are

$$C_1 e^{\lambda t}, \quad C_2 t e^{\lambda t}, \quad \ldots, \quad C_m t^{m-1} e^{\lambda t} \quad .$$

Example Find the general solution to the ODE

$$\frac{d^4 y}{dt^4} + 5\frac{d^3 y}{dt^3} + 9\frac{d^2 y}{dt^2} + 7\frac{dy}{dt} + 2y = 0 \quad .$$

Characteristic equation: $\lambda^4 + 5\lambda^3 + 9\lambda^2 + 7\lambda + 2 = 0 \implies$

$$(\lambda+2)(\lambda+1)^3 = 0 \implies$$

$$\lambda = -2, -1, -1, -1 \quad .$$

Since $f(t)=0$, $y(t) = y_h(t)$ and

$$y(t) = C_1 e^{-2t} + C_2 e^{-t} + C_3 t e^{-t} + C_4 t^2 e^{-t} \quad //$$

Particular solution

The particular solution $y_p(t)$ can be found with various methods. We consider the method of undermined coefficients in which a form of the solution is guessed based on the form of the forcing function $f(t)$.

This guess is checked by subsitition into (*). For various forms of forcing functions, the table gives "assumed" or guessed forms of $y_p(t)$.

$y_p(t)$ assumptions for method of undetermined coefficients

term in $u(t)$	assumed form for $y_p(t)$	test value
k	K_1	0
kt^n (n =1,2,...)	$K_n t^n + K_{n-1} t^{n-1} + ... + K_1 t + K_0$	0
$ke^{\lambda t}$	$K_1 e^{\lambda t}$	λ
$ke^{j\omega t}$	$K_1 e^{j\omega t}$	$j\omega$
$k\cos(\omega t)$	$K_1\cos(\omega t) + K_2\sin(\omega t)$	$j\omega$
$k\sin(\omega t)$	$K_1\cos(\omega t) + K_2\sin(\omega t)$	$j\omega$

We must take care not to guess a $y_p(t)$ that is also in the homogeneous solution $y_h(t)$. We can use the last column of the table to test this. If the "test value" in the table is equal to a characteristic polynomial root of multiplicity m, then the corresponding guessed solution must be multiplied by t^m.

Example Find a particular solution for the ODE

$$\dot{y} + 3y = e^{-3t}.$$

Characteristic equation: $\lambda + 3 = 0 \implies \lambda = -3$, $y_h(t) = K_1 e^{-3t}$.

Test value: $-3 \implies$ guess: $y_p(t) = K_2 t e^{-3t}$. Check:

$$K_2(e^{-3t} - 3t e^{-3t}) + 3K_2 t e^{-3t} = e^{-3t}$$
$$(K_2(1-3t) + 3K_2 t)e^{-3t} = e^{-3t}$$
$$K_2 e^{-3t} = e^{-3t}$$
$$\Rightarrow K_2 = 1.$$

$$y_p(t) = te^{-3t}$$

General solution

As stated above, the general solution is $y(t) = y_h(t) + y_p(t)$. The general solution still contains unspecified coefficients from the homogeneous solution. These are specified by applying the initial conditions to find the specific solution.

The following procedure gives all solutions for an ODE of the form under consideration.

1. Find the homogeneous solution $y_h(t)$.
2. Find the particular solution $y_p(t)$.
3. The general solution is $y(t) = y_h(t) + y_p(t)$.
4. Find the specific solution $y_s(t)$ by applying the initial conditions to $y(t)$.

Example Find the solution to the ODE

$$\tau \frac{dy}{dt} + y = A \cos \omega t \quad \text{with initial condition } y(0) = 0.$$

Homogeneous sol'n: $\tau\lambda + 1 = 0 \Rightarrow \lambda = -1/\tau \Rightarrow y_h(t) = Ce^{-t/\tau}$.
Particular sol'n: guess $y_p(t) = k_1 \cos \omega t + k_2 \sin \omega t$. Check:

$$\tau(-k_1\omega \sin \omega t + k_2\omega \cos \omega t) + k_1 \cos \omega t + k_2 \sin \omega t = A \cos \omega t \Rightarrow$$

$$-\tau k_1 \omega + k_2 = 0 \quad \text{and} \quad \tau k_2 \omega + k_1 = A$$

$$\Rightarrow k_2 = \tau k_1 \omega \qquad \Rightarrow \tau^2\omega^2 k_1 + k_1 = A \Rightarrow k_1 = \frac{A}{1+\tau^2\omega^2}$$

$$\Rightarrow k_2 = \frac{\tau\omega A}{1+\tau^2\omega^2}$$

Therefore, $y_p(t) = \frac{A}{1+\tau^2\omega^2}(\cos \omega t + \tau\omega \sin \omega t) = B \cos(\omega t + \phi)$,
where $B \equiv \frac{A}{1+\tau^2\omega^2}\sqrt{1+\tau^2\omega^2} = \frac{A}{\sqrt{1+\tau^2\omega^2}}$ and $\phi \equiv -\tan^{-1}\left(\frac{\tau\omega}{1}\right)$.

The general solution is $y(t) = y_h(t) + y_p(t)$. The specific sol'n is given by: $0 = Ce^0 + B\cos(0 + \phi) \Rightarrow C = -B\cos\phi \Rightarrow$

$$y_s(t) = -B\cos\phi\, e^{-t/\tau} + B\cos(\omega t + \phi).$$